FINITE-RANGE GRAVITY AND ITS ROLE IN GRAVITATIONAL WAVES, BLACK HOLES AND COSMOLOGY

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Theoretical considerations of fundamental physics, as well as certain cosmological observations, persistently point out to permissibility, and maybe necessity, of macroscopic modifications of the Einstein's general relativity. The field-theoretical formulation of general relativity helped us to identify the phenomenological seeds of such modifications. They take place in the form of very specific mass-terms, which appear in addition to the field-theoretical analog of the usual Hilbert-Einstein Lagrangian. We derive and study exact non-linear equations of the theory, along with its linear approximation. We interpret the added terms as masses of the $spin-2$ and $spin-0$ gravitons. The arising finite-range gravity is a fully consistent theory, which smoothly approaches general relativity in the massless limit, that is, when both masses tend to zero and the range of gravity tends to infinity. We show that all local weak-field predictions of the theory are in perfect agreement with the available experimental data. However, some other conclusions of the non-linear massive theory are in a striking contrast with those of general relativity. We show in detail how the arbitrarily small mass-terms eliminate the black hole event horizon and replace a permanent power-law expansion of a homogeneous isotropic universe with an oscillatory behaviour. One variant of the theory allows the cosmological scale factor to exhibit an "accelerated expansion" instead of slowing down to a regular maximum of expansion. We show in detail why the traditional, Fierz-Pauli, mass-term is unacceptable as being in conflict not only with the static-field experiments but also with the available indirect gravitational-wave observations.

I. INTRODUCTION

Presently, there seems to be no pressing need in devising theories of gravitation, alternative to the existing Einstein's general relativity. General relativity (GR) is an internally consistent theory, and it has passed all the performed experimental tests with flying colors [1], [2], [3], [4], [5]. And yet, there are some clouds on the horizon. On the theoretical side, the M/string theory persistently points out to possible macroscopic modifications of GR, particularly in the form of various "mass-terms". On the observational side, there exists some discomfort in understanding the large-scale structure and evolution of the Universe, including some indications to the possibility of its present "accelerated expansion". So far, theorists enjoy playing with the cosmological Λ-term and various highly speculative forms of matter, but the credibility of these models can soon be exhausted. The old question arises again, whether there do exist well-motivated consistent alternative theories of macroscopic gravity, with non-trivial observational consequences.

At the first sight, general relativity is an isolated theory with no immediate neighbours. In particular, it seems that GR cannot be modified without raising the order of differential field equations. Indeed, in the geometrical formulation of GR, which operates with the curved space-time metric tensor $g_{\mu\nu}$, there is no structure that can be added to the usual Hilbert-Einstein Lagrangian. The only possibility is the Λ-term: $\sqrt{-g}\Lambda$, but this structure can be included in the definition of GR, and we know all the phenomenological consequences of the Λ-term, and in any case the Λ-term is not a "mass-term". The situation changes drastically when one looks at GR from the field-theoretical perspective. The old remark of Feynman [6] on the intrinsic value of equivalent formulations of a fundamental theory proves to be very profound. One gets the possibility to analyse the problems which otherwise could not be even properly formulated. We believe that general relativity does indeed contain the seeds of its own modification, and the field-theoretical formulation of GR helped us to identify these seeds. The modification of GR, which looks almost unavoidable from the viewpoint of the field-theoretical approach, leads to the appearance of very specific mass-terms. The resulting theory is a fully consistent finite-range gravitational theory. General relativity is a smooth limit of this theory when the range of gravity tends to infinity. The theory is in perfect agreement with all local weak-field experiments, such as experiments in the Solar system, and satisfies the requirement, formulated long ago [7], of "physical continuity". However, some other consequences of the theory are truly striking. It is surprising to see that some of the crucial conclusions of GR are so much vulnerable to pretty innocent modifications of GR. For instance, the existence of a black hole event horizon, and a permanent power-law expansion of the matter-dominated Universe, get invalidated by the arbitrarily small mass-terms. We introduce and explain this finite-range gravitational theory in the present paper.

The fundamental quantity in the field-theoretical GR is a symmetric second-rank tensor field $h^{\mu\nu}(x^\alpha)$. The gravitational field $h^{\mu\nu}(x^\alpha)$ is defined in a flat space-time with the line-element

$$\mathrm{d}\sigma^2 = \gamma_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{1}$$

The curvature tensor constructed from $\gamma_{\mu\nu}(x^\alpha)$ is identically zero:

$$\breve{R}_{\alpha\beta\mu\nu}(\gamma_{\rho\sigma}) = 0. \tag{2}$$

In flat space-time, one is always free to choose Lorentzian coordinates, in which case Eq. (1) takes on the Minkowski form

$$\mathrm{d}\sigma^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{3}$$

The flat space-time is not a choice of some artificial "prior geometry", but is a reflection of experimental facts. As far as the present-day physics knows, the intervals of space and durations of time, in absence of all fields including gravity, satisfy the relationships of the Minkowski 4-dimensional interval (3). If there existed any observational evidence to something different, we would have started from a different metric.

The Lagrangian of the field-theoretical GR depends on the gravitational field variables $h^{\mu\nu}(x^\alpha)$ and their first derivatives. (We present more details in Sec. II) The variational principle gives rise to the dynamical field equations, which are fully equivalent to the Einstein equations. The transition to the geometrical formulation of GR proceeds through the introduction of the tensor $g^{\mu\nu}(x^\alpha)$ and the inverse tensor $g_{\mu\nu}(x^\alpha)$: $g^{\mu\rho}g_{\nu\rho} = \delta^\mu_\nu$. The quantities $g^{\mu\nu}$ are calculable from the gravitational field variables $h^{\mu\nu}$ and the metric tensor $\gamma^{\mu\nu}$ according to the rule

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}) \ , \tag{4}$$

where $g = det|g_{\mu\nu}|$, $\gamma = det|\gamma_{\mu\nu}|$, and $\gamma^{\mu\rho}\gamma_{\nu\rho} = \delta^\mu_\nu$. The tensor density $\sqrt{-g}g^{\mu\nu}$ participates in the matter Lagrangian, realizing the universal coupling of gravity to all other fields, but apart of that, it is simply a short-hand notation for the quantity in the right-hand-side (r.h.s.) of Eq. (4). In the geometrical formulation of GR, tensor $g_{\mu\nu}(x^\alpha)$ is interpreted as the metric tensor of a curved space-time:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{5}$$

In terms of $g_{\mu\nu}$, the field equations acquire the familiar form of the geometrical Einstein's equations. From the viewpoint of the field-theoretical formulation, the tensor $g_{\mu\nu}(x^\alpha)$ is the effective metric tensor; it defines the intervals of space and time measured in the presence of the universal gravitational field $h^{\mu\nu}(x^\alpha)$. The field-theoretical approach to GR has a long and fruitful history. For a sample of references, see [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], including a history review [18], and many papers cited therein.

It was shown [19] that the gravitational Lagrangian of the field-theoretical GR must include, in addition to the field-theoretical analog of the Hilbert-Einstein term, the extra term

$$\sqrt{-\gamma}\left[-\frac{1}{4}\breve{R}_{\alpha\rho\beta\sigma}(h^{\alpha\beta}h^{\rho\sigma} - h^{\alpha\sigma}h^{\rho\beta})\right]. \tag{6}$$

This term does not affect the field equations, but is needed for the variational derivation of the gravitational energy-momentum tensor $t^{\mu\nu}$. The variational (metrical) energy-momentum tensor is a response of the physical system to variations of the metric tensor $\gamma_{\mu\nu}$, caused by arbitrary coordinate transformations. Obviously, such variations of the metric tensor should obey the constraint (2). The variational procedure incorporates the constraint (2) by adding to the Lagrangian an extra term: $\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}$, where $\Lambda^{\alpha\beta\rho\sigma}$ are undetermined Lagrange multipliers. The constraint (2) has to be enforced at the end of the variational derivation of the field equations and the energy-momentum tensor. It has been proven [19] that the Lagrange multipliers must have the unique form

$$\Lambda^{\mu\nu\alpha\beta} = -\frac{1}{4}(h^{\alpha\beta}h^{\mu\nu} - h^{\alpha\nu}h^{\beta\mu}),$$

in order for the derived energy-momentum tensor $t^{\mu\nu}$ to satisfy all the necessary mathematical and physical requirements, including the absence of second derivatives of the field variables in the $t^{\mu\nu}$.

As was explained above, the quantity $\breve{R}_{\alpha\rho\beta\sigma}$ in Eq. (6) is the curvature tensor of a flat space-time. If it were something other than that, the theory would not be GR. However, it is natural to assume that the Lagrangian may also include an additional term similar to (6), but where the quantity $\breve{R}_{\alpha\rho\beta\sigma}$ is the curvature tensor of an abstract space-time with a constant non-zero curvature. Space-times of constant curvature are as symmetric as flat space-time, but contain a parameter K with dimensionality of $[length]^{-2}$:

$$\breve{R}_{\alpha\beta\mu\nu} = K(\gamma_{\alpha\mu}\gamma_{\beta\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu}). \tag{7}$$

If one uses (7) in (6), the generated additional term in the Lagrangian is

$$\sqrt{-\gamma}\frac{K}{2}(h^{\alpha\beta}h_{\alpha\beta} - h^2). \tag{8}$$

Clearly, the new theory is not GR, but what this theory is ? Quite surprisingly, one recognizes in (8) the Fierz-Pauli [20] mass-term. Having discovered that the structure (6) generates mass-terms, we have asked about the most general form of such terms. It is easy to show that there exist only two independent quadratic combinations: $h^{\alpha\beta}h_{\alpha\beta}$ and h^2. Therefore, we arrive at a 2-parameter family of theories with the additional mass-terms in the gravitational Lagrangian:

$$\sqrt{-\gamma}\left[k_1 h^{\rho\sigma}h_{\rho\sigma} + k_2 h^2\right], \tag{9}$$

where k_1 and k_2 have dimensionality of $[length]^{-2}$. Fierz and Pauli, as well as many other authors after them, were considering the (internally contradictory) “linear gravity”, whereas in our case the tensor $h^{\mu\nu}$ is the full-fledged non-linear gravitational field. The 2-parameter class of theories with the additional mass-terms (9) is what we shall study in the present paper. We consider the mass-terms as phenomenological, even though their deep origin can be quantum-mechanical or multi-dimensional.

The structure of the paper and its conclusions are as follows.

In Sec. II, we derive exact non-linear equations for the gravitational field in absence of any matter sources. Since practically all calculations in gravitational physics are performed in geometrical language, and we will need some of the results, we introduce the notion of quasi-geometrical description of the finite-range gravity.[1] Specifically, we retain the usual presentation of the Einstein part of the equations in terms of $g_{\mu\nu}$, whereas in the massive part, which originates from (9) and cannot be written in terms of $g_{\mu\nu}$ only, we trade $h_{\mu\nu}$ for $g_{\mu\nu}$ and $\gamma_{\mu\nu}$, according to the rule (4). The important point is the symmetries of the theory. Equations of the field-theoretical GR enjoy two different symmetries. The first one (general covariance, or diffeomorphism) is the freedom to use arbitrary coordinates and the associated transformations of, both, the metric tensor $\gamma^{\mu\nu}$ and the field tensor $h^{\mu\nu}$. The second symmetry is the freedom to use the (true) gauge transformations, which do not touch coordinates and the metric tensor, but transform the field variables only [16]. It is this second symmetry that gets violated by the mass-terms, while the first symmetry survives.

In Sec. III, we formulate exact equations for the gravitational field in the presence of matter sources. Again, we are often using the quasi-geometrical description. This means, in particular, that in the matter part of the field equations we retain the geometrical energy-momentum tensor $T_{\mu\nu}$, i.e. the matter energy-momentum tensor defined as the variational derivative of the matter Lagrangian with respect to $g^{\mu\nu}$, as opposed to the field-theoretical energy-momentum tensor $\tau_{\mu\nu}$, defined as the variational derivative of the matter Lagrangian with respect to $\gamma^{\mu\nu}$. The content of Sec. III will be needed in Section VII and, partially, in Section V.

In Sec. IV, we discuss the linearised approximation of the theory and give physical interpretation to the parameters k_1 and k_2. In accord with the analysis of Ogievetsky and Polubarinov [21], and Van Dam and Veltman [22], these parameters give rise to two fundamental masses: the mass m_2 of the $spin-2$ graviton, and the mass m_0 of the $spin-0$ graviton. Strictly speaking, the corresponding wave-equations contain two fundamental lengths, rather than two fundamental masses. Concretely, the equations contain two parameters, α^2 and β^2, with dimensionalities of $[length]^{-2}$:

$$\alpha^2 = 4k_1, \qquad \beta^2 = -2k_1\frac{k_1 + 4k_2}{k_1 + k_2}, \tag{10}$$

but α and β can be thought of as inverse Compton wavelengths of the two gravitons with the masses

$$m_2 = \frac{\alpha\hbar}{c}, \qquad m_0 = \frac{\beta\hbar}{c}. \tag{11}$$

[1] Geometry in physics, like communism in politics, is not dangerous, if introduced in well-measured doses.

The interpretation of the free parameters in terms of masses implies that α^2 and β^2 are strictly positive quantities. However, the Lagrangian itself does not require this restriction, and we will exploit this freedom in the cosmological Section VII.

One very special choice of the parameters k_1 and k_2 is $k_2 = -k_1$. This choice of the parameters brings the Lagrangian (9) to the Fierz-Pauli form (8). It is this case that has led to a lively debate on the unacceptability of a "massive graviton". Although the Lagrangian (8) itself does smoothly vanish in the limit $k_1 \to 0$, the corresponding solutions and local weak-field physical predictions (for instance, the deflection angle of light propagating in the gravitational field of the Sun) do not approach those of GR. In other words, this particular massive theory disagrees with the original massless theory even in the limit of vanishingly small mass m_2 and, hence, in the limit of arbitrarily long Compton wavelength $1/\alpha$. The finite, and independent of the mass m_2, difference in local predictions became known as the Van Dam-Veltman-Zakharov discontinuity [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33]. This puzzling conclusion about discontinuity is described [34] as something that seems counter-intuitive to certain physicists. We have to confess that the usual presentation of this conclusion seems counter-intuitive to us, as well. We believe that the issue should be looked upon from a different angle. When taking the massless limit of the massive theory, one should do what the logic requires to do, namely, to send both masses to zero. Then, both Compton wavelengths tend to infinity, and one recovers, as expected, the local weak-field predictions of GR. If, instead, one takes $k_2 = -k_1$ (whatever the motivations behind this choice might be), the mass m_0 becomes infinitely large (see Eqs. (10), (11)) and the corresponding Compton wavelength $1/\beta$ is being sent to zero. Any local experiment is now supposed to be performed at scales much larger than one of the characteristic lengths, $1/\beta$. In this situation, the deviations from GR should be expected on the grounds of physical intuition. There is no wonder that the subsequent limit $1/\alpha \to \infty$ does not cure these deviations. This situation may look like a counter-intuitive discontinuity.

To explore the difference in local predictions, there is no need to propagate light in the Solar system. It is sufficient to consider the geodesic deviation equation for free test bodies separated by small distances. We do this study below in the paper. In particular, the geodesic deviation equation illustrates the difference between GR and finite-range gravity in the domain of gravitational-wave predictions.

In Sec. V, we study weak gravitational waves. Certain modifications of GR are well anticipated. In the field-theoretical GR, the $spin-0$ gravitational waves (represented by the trace $h = h^{\mu\nu}\eta_{\mu\nu}$) exist as gauge solutions. They contribute neither to the gravitational energy-momentum tensor $t^{\mu\nu}$, nor to the deformation pattern of a ring of test particles in the geodesic deviation equation. The same is true for the $helicity-0$ polarization state (represented by the spatial trace $h^{ij}\eta_{ij}$) of the $spin-2$ graviton. In the finite-range gravity, as one could expect, both these degrees of freedom become essential. They provide additional contributions to the energy-momentum flux carried by the gravitational wave, and extra components of motion to the test particles. Gravitational wave solutions and physical predictions of GR are fully recovered, though, in the massless limit $\alpha \to 0$, $\beta \to 0$ of the theory. We show that the Fierz-Pauli case is very peculiar and unacceptable. Even in the limit of $\alpha \to 0$, there remains a nonvanishing "common mode" motion of test particles in the plane of the wave front. The extra component of motion is accounted for by the corresponding additional flux of energy from the source; typically, of the same order of magnitude as the GR flux. This analysis, together with the Solar system arguments, leads to the important conclusion. Whatever the sophisticated "brane world" motivations of the M/string theory may be, if they lead to the phenomenological mass-term of the Fierz-Pauli type, the corresponding variant of the theory should be rejected as being in conflict with the static-field experiments and with the already available indirect gravitational-wave observations of binary pulsars. We do not think that this conclusion can be invalidated by any "non-perturbative effects".

The fully non-linear finite-range gravity is considered in the next two Sections. In Sec. VI, we analyse static spherically-symmetric solutions. First, we summarise the situation in the weak-field approximation. We consider intermediate scales, that is, the Schwarzschild distances R which are much larger than $2GM/c^2$ (where M is the Schwarzschild mass), but much smaller than $1/\alpha$ and $1/\beta$. We demonstrate that the GR solutions and physical predictions are recovered in the massless limit $\alpha \to 0$, $\beta \to 0$. With the help of the geodesic deviation equation, we confirm the observational unacceptability of the Fierz-Pauli coupling.

The main thrust of Sec. VI is the non-linear (would-be black hole) solutions. The case of arbitrary relationship between α and β is difficult to analyse in full generality. The equations are somewhat simpler when the masses are assumed to be equal, i.e. $\alpha = \beta$. We call this choice the Ogievetsky-Polubarinov (OP) case. We analyse this case in great detail, and present more general considerations whenever possible, demonstrating that the qualitative conclusions remain valid for $\alpha \neq \beta$. A single dimensionless parameter in the OP case is αM (using $G = 1$ and $c = 1$) which is supposed to be a very small number. Combining analytical and numerical techniques, we demonstrate that the solution of the massive theory is practically indistinguishable from that of GR for all R sufficiently larger than $2M$, but, obviously, smaller than $1/\alpha$. As expected, for R larger than $1/\alpha$, the solution takes on the form of the Yukawa-type potentials; this is why the theory is called finite-range gravity. However, the massive solution strongly deviates from that of GR not only at very large distances, but also in the vicinity of $R = 2M$. This is a consequence of the non-linear character of the field equations. The hypersurface $R = 2M$ is the location of the (globally defined) event horizon of the Schwarzschild black hole in GR. We carefully explore the vicinity of $R = 2M$, as well as $0 \leq R < 2M$, the region that would have been the interior of the Schwarzschild black hole. We show that the smaller the parameter αM, the closer to $R = 2M$ one can descend (from large R) along the essentially Schwarzschild solution. We show that the deviations from GR near $R = 2M$ are so radical that the event horizon does not form, and the solution smoothly continues to the region $R < 2M$. The further continuation of the solution terminates at $R = 0$, where the curvature singularity develops. Since the αM can be extremely small, the redshift of the photon emitted at $R = 2M$ can be extremely large, but it remains finite. In contrast to GR, the infinite redshift is reached at the singularity $R = 0$, and not at $R = 2M$. The conclusion of this study is quite dramatic. In the astrophysical sense, the resulting solution still looks like a black hole; in the region of space just outside the $R = 2M$, the gravitational field is practically indistinguishable from the Schwarzschild solution. However, all conclusions that rely specifically on the existence of the black hole event horizon, are likely to be abandoned. It is very remarkable and surprising that the phenomenon of black hole should be so unstable with respect to the inclusion of the tiny mass-terms (9), whose Compton wavelengths can exceed, say, the present-day Hubble radius.

Section VII is devoted to cosmological solutions for homogeneous isotropic universe. As matter sources, we consider simplest models of perfect fluid with a fixed equation of state. First, we show that if the mass of the $spin-0$ graviton is zero, i.e. $\beta^2 = 0$, the cosmological solutions of the massive theory are exactly the same as those of GR, independently of the mass of the $spin - 2$ graviton, that is, independently of the value of α^2. This result could be expected due to the highest spatial symmetry of the problem under consideration; the $spin - 2$ degrees of freedom have no chance to reveal themselves. Then, we proceed to cases with $\beta^2 \neq 0$. Since we prefer to deal with technically simple equations, we consider a particular case $4\beta^2 = \alpha^2$. This case is studied in full details, but we also show that the qualitative results are general and are valid for $4\beta^2 \neq \alpha^2$. Combining analytical approximations and numerical calculations, we demonstrate that the massive solution has a long interval of evolution where it is practically indistinguishable from the Friedmann solution of GR. However, the deviations from GR are dramatic at very early times and very late times. The unlimited expansion is being replaced by a regular maximum of the scale factor, whereas the singularity is being replaced by a regular minimum of the scale factor. The smaller β, the higher maximum and the deeper minimum. In other words, astonishingly, the arbitrarily small mass-terms (9) give rise to the oscillatory behaviour of the cosmological scale factor.

Following the logic of interpretation of the theory in terms of masses, we assume in the bulk of the paper that the signs of α^2 and β^2 are positive. However, as mentioned above, the general structure of the Lagrangian (6) does not imply this. It is interesting to observe that if we allow α^2 and β^2 to be negative (which would probably require to think of the massive gravitons in terms of "tachyons"), the late time evolution of the scale factor exhibits an "accelerated expansion", instead of slowing down towards the maximum. This behaviour of the scale factor is similar to the one governed by a positive cosmological Λ-term. The physical significance of this result is presently unclear, but the problem deserves further study.

Some technical details of the paper are relegated to the Appendix.

II. SOURCE-FREE GRAVITATIONAL FIELD

The gravitational contribution S^g to the total action is

$$S^g = \frac{1}{c} \int L^g \, d^4x.$$

The gravitational Lagrangian density L^g consists of two parts - the GR part and the massive part:

$$L^g = L^g_{GR} + L^g_{mass}. \tag{12}$$

As was explained in Introduction, the GR part itself consists of two terms [19], which are i) the field-theoretical analog of the Hilbert-Einstein Lagrangian and ii) the term incorporating the constraint (2):

$$L^g_{GR} = -\frac{\sqrt{-\gamma}}{2\kappa} \left\{ \frac{1}{2} \Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau} h^{\rho\sigma}{}_{;\tau} h^{\alpha\beta}{}_{;\omega} - \frac{1}{4} (h^{\rho\sigma} h^{\alpha\beta} - h^{\alpha\sigma} h^{\beta\rho}) \breve{R}_{\alpha\rho\beta\sigma} \right\}, \tag{13}$$

where $\kappa = 8\pi G/c^4$. The field-theoretical analog of the Hilbert-Einstein term has the form similar to the kinetic energy of classical mechanics; the Lagrangian is manifestly quadratic in the generalised velocities $h^{\mu\nu}{}_{;\tau}$. [We remind the reader that the raising and lowering of indeces of the field $h^{\mu\nu}$, and its covariant differentiation denoted by a semicolon ";", are performed with the help of the metric tensor $\gamma_{\mu\nu}$ and its Christoffel symbols $C^{\alpha}_{\rho\sigma}$.] The generalised coordinates $h^{\mu\nu}$ are present only in the tensor $\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau}$. For the reference, we reproduce here the compact expression of this tensor, but we refer to [19] for details,

$$\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\tau\omega} = \frac{1}{4}\frac{\sqrt{-\gamma}}{\sqrt{-g}}[(\delta^\tau_\mu\delta^\pi_\nu + \delta^\tau_\nu\delta^\pi_\mu)(\delta^\omega_\rho\delta^\lambda_\sigma + \delta^\omega_\sigma\delta^\lambda_\rho)g_{\pi\lambda} - g^{\tau\omega}(g_{\mu\rho}g_{\nu\sigma} + g_{\nu\rho}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma})] \ . \tag{14}$$

As was explained in Introduction, the massive part of the Lagrangian is given by

$$L^g_{mass} = -\frac{\sqrt{-\gamma}}{2\kappa}\left\{k_1 h^{\rho\sigma}h_{\rho\sigma} + k_2 h^2\right\}, \tag{15}$$

where, obviously, $h \equiv h^{\alpha\beta}\gamma_{\alpha\beta}$.

Having defined the gravitational Lagrangian, we are in the position to derive the source-free field equations:

$$\frac{\delta L^g}{\delta h^{\alpha\beta}} = \frac{\partial L^g}{\partial h^{\alpha\beta}} - \left(\frac{\partial L^g}{\partial h^{\alpha\beta}{}_{;\sigma}}\right)_{;\sigma} = 0 \ . \tag{16}$$

Certainly, the GR part alone generates the equations completely equivalent to the Einstein equations:

$$\frac{1}{2}\left[(\gamma^{\mu\nu} + h^{\mu\nu})(\gamma^{\alpha\beta} + h^{\alpha\beta}) - (\gamma^{\mu\alpha} + h^{\mu\alpha})(\gamma^{\nu\beta} + h^{\nu\beta})\right]_{;\alpha;\beta} = \kappa t^{\mu\nu}, \tag{17}$$

where $t^{\mu\nu}$ is the gravitational energy-momentum tensor satisfying all the necessary requirements [19]. [2] To write the equation (17) in the geometrical language, one composes the combination of equations (16) by multiplying them with the factor $\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}$, and uses the relationship (4) (for details, see [19]). As a result, one arrives at

$$G_{\mu\nu} = 0, \tag{18}$$

where $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{19}$$

and $R_{\mu\nu}$ is the Ricci tensor constructed from $g_{\mu\nu}$ in the usual manner.

The massive part of L^g makes its own contribution to the field equations,

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^g_{mass}}{\delta h^{\mu\nu}} = 2k_1 h_{\mu\nu} + 2k_2\gamma_{\mu\nu}h \ . \tag{20}$$

The field equations (18) get modified. Taking into account (20) and repeating the steps described above for the GR case, one arrives at the source-free equations of the finite-range theory:

$$G_{\mu\nu} + M_{\mu\nu} = 0, \tag{21}$$

where

$$M_{\mu\nu} \equiv \left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\right)(2k_1 h_{\alpha\beta} + 2k_2\gamma_{\alpha\beta}h) \, . \tag{22}$$

Replacing, with the help of (4), $h^{\mu\nu}$ in favour of $g^{\mu\nu}$ and $\gamma^{\mu\nu}$, one obtains the quasi-geometric form of $M_{\mu\nu}$:

$$M_{\mu\nu} = 2\gamma_{\rho\alpha}\gamma_{\sigma\beta}\left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\right)\left[k_1\left(\frac{\sqrt{-g}}{\sqrt{-\gamma}}g^{\rho\sigma} - \gamma^{\rho\sigma}\right) + k_2\gamma^{\rho\sigma}\left(\frac{\sqrt{-g}}{\sqrt{-\gamma}}g^{\tau\psi}\gamma_{\tau\psi} - 4\right)\right] . \tag{23}$$

[2]It appears that the attitude towards the field-theoretical GR is approaching the last phase of a quite usual response, when the dialog begins with the objection “this is impossible and cannot be true”, goes through “this is interesting but has not been proven”, and finishes with the reassuring “this is correct and wonderful, and I have done this long ago”.

Thus, in the source-free case, we have to solve equations (21), (23), instead of the GR equations (18). The choice of coordinates and, hence, the form of the metric tensor $\gamma_{\mu\nu}$, is entirely in our hands. In what follows, we will be using Lorentzian coordinates (3) or, where convenient, spatially-spherical coordinates,

$$\mathrm{d}\sigma^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2). \tag{24}$$

One difference between Eq. (21) and Eq. (18) is apparent. Let us denote by a stroke "|" the covariant derivative defined with the help of the effective metric tensor $g_{\mu\nu}$ and its Christoffel symbols $\Gamma^{\alpha}_{\rho\sigma}$. Then, the Bianchi identities read

$$g^{\mu\alpha}G_{\alpha\nu|\mu} \equiv G^{\mu}{}_{\nu|\mu} \equiv 0. \tag{25}$$

In other words, this particular combination of equations (18) is satisfied identically. This is not so in the case of equations (21). Applying the same differentiation, one arrives at the non-trivial consequences of equations (21):

$$g^{\mu\alpha}M_{\alpha\nu|\mu} \equiv M^{\mu}{}_{\nu|\mu} = 0. \tag{26}$$

Although equations (26) are merely the consequences of the full set of equations (21), and therefore contain no new information, it proves convenient, as will be shown below, to use them instead of some members of the original set of equations (21).

It is interesting that Eq. (26) can also be written in the totally equivalent form, which employs the field variables $h^{\mu\nu}$ and the metric tensor $\gamma^{\mu\nu}$:

$$\mathcal{M}^{\mu\nu}{}_{;\nu} = 0, \tag{27}$$

where

$$\mathcal{M}^{\mu\nu} \equiv 2k_1h^{\mu\nu} - (k_1 + 2k_2)\gamma^{\mu\nu}h + 2k_1h^{\nu\beta}h^{\mu}{}_{\beta} + 2k_2h^{\mu\nu}h - \frac{1}{2}\gamma^{\mu\nu}(k_1h^{\alpha\beta}h_{\alpha\beta} + k_2h^2). \tag{28}$$

We will prove the equivalence of (26) and (27) in the Appendix. The representation (27) will be especially helpful in the cosmological Sec. VII.

III. GRAVITATIONAL FIELD WITH MATTER SOURCES

The total action in the presence of matter sources is

$$S = \frac{1}{c}\int (L^g + L^m)\, d^4x,$$

where L^m is the Lagrangian density for matter fields. L^m includes also the interaction of matter fields with the gravitational field. One or several matter fields are denoted by ϕ_A, where A is some general index. We assume the universal coupling of all matter fields to the gravitational field. Specifically, we assume that L^m depends on the gravitational field variables $h^{\mu\nu}$ in a particular manner, namely, through the combination $\sqrt{-g}g^{\mu\nu}$:

$$L^m = L^m\left[\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}); (\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}))_{,\alpha}; \phi_A; \phi_{A,\alpha}\right]. \tag{29}$$

The adopted coupling of matter to gravity is exactly the same as in GR. Therefore the matter field equations,

$$\frac{\delta L^m}{\delta\phi_A} = 0, \tag{30}$$

are also exactly the same as in GR.

We can now derive the gravitational field equations with matter sources,

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^{tot}}{\delta h^{\mu\nu}} = -\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta h^{\mu\nu}} - \frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta h^{\mu\nu}} = 0. \tag{31}$$

Everything what comes out of L^g is already known. The contribution of L^m to the gravitational field equations can be worked out by taking the advantage of the specific form of Eq. (29). Indeed, one can write

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta h^{\mu\nu}} = 2\frac{\delta L^m}{\delta(\sqrt{-g}g^{\rho\sigma})} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}. \tag{32}$$

where $T_{\mu\nu}$ is the (geometrical) energy-momentum tensor of the matter. It is defined as the variational derivative of L^m with respect to $g^{\mu\nu}$:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta L^m}{\delta g^{\mu\nu}}.$$

Multiplying the field equations (31) by the factor $\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}$ and rearranging the terms, we arrive at the gravitational field equations of the finite-range gravity with matter sources:

$$G_{\mu\nu} + M_{\mu\nu} = \kappa T_{\mu\nu}. \tag{33}$$

Let us now discuss the consequences of these equations related to the Bianchi identities $G^\mu{}_{\nu|\mu} \equiv 0$. It is important to remember that the conservation equations

$$T^\mu{}_{\nu|\mu} = 0 \tag{34}$$

are satisfied as soon as the matter equations of motion (30) are satisfied. In other words, equations (34) are satisfied independently of the gravitational field equations. Therefore, taking the |-covariant divergence of Eq. (33) and assuming that Eq. (30) are fulfilled, we obtain equations (26) and (27), i.e. exactly the same equations as in the source-free case.

IV. LINEARISED THEORY OF THE SOURCE-FREE GRAVITATIONAL FIELD

The proper physical interpretation of the free parameters k_1 and k_2 is revealed from the linearised field equations. The linearisation means that the quantities $h^{\mu\nu}$ are regarded small, and only terms linear in $h^{\mu\nu}$ are retained in the field equations. To find the linear version of Eq. (21) one can use the linear version of Eq. (4):

$$g^{\mu\nu} \approx \gamma^{\mu\nu} + (h^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}h). \tag{35}$$

It is also possible to start from Eq. (17) remembering that $t^{\mu\nu}$ is not less than quadratic in $h^{\mu\nu}$. By either route, one arrives at the linearised version of equations for the finite-range gravity:

$$\frac{1}{2}\left[h^{\mu\nu;\alpha}{}_{;\alpha} + \gamma^{\mu\nu}h^{\alpha\beta}{}_{;\alpha;\beta} - h^{\nu\alpha;\mu}{}_{;\alpha} - h^{\mu\alpha;\nu}{}_{;\alpha}\right] + \left[2k_1h^{\mu\nu} - (k_1+2k_2)\,\gamma^{\mu\nu}h\right] = 0. \tag{36}$$

The useful consequence of Eq. (36) is derived by taking the ;-covariant divergence of Eq. (36). Since the ;-covariant divergence of the GR part (first square bracket) is identically zero, we obtain

$$\left[2k_1h^{\mu\nu} - (k_1+2k_2)\,\gamma^{\mu\nu}h\right]_{;\nu} = 0. \tag{37}$$

Clearly, Eq. (37) is the linearised version of the exact equations (27).

Apparently, the first study of the two-parameter set of equations (36) has been done by Fierz and Pauli [20] who assumed that $k_2 = -k_1$. We will consider here the general case of arbitrary parameters. Following Van Dam and Veltman [22], we will start from transforming Eq. (36) to the more suggestive set of equations.

Taking the covariant divergence of (37) one derives

$$2k_1h^{\mu\nu}{}_{;\mu;\nu} = (k_1+2k_2)\,\Box h, \tag{38}$$

where the symbol $\Box$ denotes the d'Alembert operator in arbitrary (in general, curvilinear) coordinates: $\Box Z = \gamma^{\alpha\beta}Z_{;\alpha;\beta}$. Taking the trace of Eq. (36) one derives

$$\Box h + 2h^{\alpha\beta}{}_{;\alpha;\beta} - 4(k_1+4k_2)h = 0. \tag{39}$$

Combining Eq. (38) and Eq. (39) in order to exclude $h^{\alpha\beta}{}_{;\alpha;\beta}$, one obtains the equation for the trace h:

$$\Box h - 2k_1\frac{k_1+4k_2}{k_1+k_2}h = 0. \tag{40}$$

Obviously, when writing down the equation (40), we assume that $k_2 \neq -k_1$. Otherwise, i.e. in the Fierz-Pauli case $k_1 + k_2 = 0$, the full set of equations (36) is equivalent to

$$h = 0, \quad h^{\mu\nu}{}_{;\nu} = 0, \quad \Box h^{\mu\nu} + 4k_1 h^{\mu\nu} = 0. \tag{41}$$

We shall discuss the observational consequences of the Fierz-Pauli theory in Sections V and VI. Meanwhile, we shall return to the general case $k_2 \neq -k_1$.

It is convenient to introduce the quantity $H^{\mu\nu}$ according to the relationship

$$H^{\mu\nu} = h^{\mu\nu} - \frac{k_1 + k_2}{3k_1}\gamma^{\mu\nu} h - \frac{k_1 + k_2}{6k_1^2} h^{;\mu;\nu} + \frac{k_1 + k_2}{12k_1^2}\gamma^{\mu\nu}\Box h. \tag{42}$$

The trace and the covariant divergence of $H^{\mu\nu}$ vanish due to the equations (37) and (40),

$$\gamma_{\mu\nu} H^{\mu\nu} = 0, \quad H^{\mu\nu}{}_{;\nu} = 0. \tag{43}$$

In terms of $H^{\mu\nu}$, and taking into account (37) and (40), equations (36) transform to

$$\Box H^{\mu\nu} + 4k_1 H^{\mu\nu} = 0. \tag{44}$$

We now introduce α^2 and β^2 according to the definitions (10). Then, the full set of equations (36) is equivalent to

$$2k_1 h^{\mu\nu}{}_{;\nu} - (k_1 + 2k_2)\, h^{;\mu} = 0, \tag{45}$$

$$\Box h + \beta^2 h = 0, \tag{46}$$

$$\Box H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 0. \tag{47}$$

The wave-like form of the resulting equations (47) and (46) justifies the interpretation of the parameters k_1 and k_2,

$$k_1 = \frac{\alpha^2}{4}, \qquad k_2 = -\frac{\alpha^2(\alpha^2 + 2\beta^2)}{8(2\alpha^2 + \beta^2)},$$

in terms of masses (11). The constraints (43) allow one to associate the tensor field $H^{\mu\nu}$ with the *spin* $-$ 2 graviton, whereas the scalar quantity h can be associated with the *spin* $-$ 0 graviton.

V. WEAK GRAVITATIONAL WAVES

We shall start from a brief summary of gravitational waves in GR, and we shall do this in the framework most appropriate for the further comparison with the case of massive theory. In this Section, it is convenient to work in Lorentzian coordinates (3). This means that the metric tensor $\gamma_{\mu\nu}$ simplifies to $\eta_{\mu\nu}$, and the ;-covariant derivative simplifies to the ordinary derivative, denoted by a comma.

A. Weak gravitational waves in general relativity.

The linearised source-free equations (36) take the form

$$h^{\mu\nu,\alpha}{}_{,\alpha} + \eta^{\mu\nu} h^{\alpha\beta}{}_{,\alpha,\beta} - h^{\nu\alpha,\mu}{}_{,\alpha} - h^{\mu\alpha,\nu}{}_{,\alpha} = 0. \tag{48}$$

A general plane-wave solution to Eq. (48) is given by

$$h^{\mu\nu} = a^{\mu\nu} e^{ik_\alpha x^\alpha} + (c^\mu q^\nu + c^\nu q^\mu - \eta^{\mu\nu} c^\alpha q_\alpha) e^{iq_\alpha x^\alpha} + c.c., \tag{49}$$

where *c.c* denotes the complex conjugate part; $k_\alpha k^\alpha = 0$; $q_\alpha q^\alpha$ can be of any sign or zero; 4 quantities c^μ are arbitrary; 10 components of the matrix $a^{\mu\nu}$ are constrained by 4 conditions:

$$a^{\mu\nu} k_\nu = 0. \tag{50}$$

Conditions (50) allow one to express a^{00} and a^{0i} in terms of 6 independent components of the spatial matrix a^{ij}:

$$a^{00} = \frac{1}{k_0^2} a^{ij} k_i k_j, \qquad a^{0i} = -\frac{1}{k_0} a^{ij} k_j. \tag{51}$$

The matrix a^{ij} itself can be written in the general form

$$a^{ij} = \tilde{a}^{ij} + b^i k^j + b^j k^i + b\eta^{ij}, \tag{52}$$

where 4 quantities b, b^i are arbitrary, while the matrix $\tilde{a}^{ij}$ has only 2 independent components (sometimes called TT-components) as it satisfies 4 conditions:

$$\tilde{a}^{ij} k_j = 0, \qquad \tilde{a}^{ij} \eta_{ij} = 0. \tag{53}$$

The gravitational energy-momentum tensor $t_{\mu\nu}$, in its lowest nonvanishing approximation, is given by

$$t_{\mu\nu} = \frac{1}{4\kappa} \left[h^{\alpha\beta}{}_{,\mu} h_{\alpha\beta,\nu} - \frac{1}{2} h_{,\mu} h_{,\nu} \right]. \tag{54}$$

Using the relationships above, one can show that the expression (54) depends only on the TT-components of the matrix a^{ij}. Specifically,

$$t_{\mu\nu} = \frac{1}{4\kappa} k_\mu k_\nu \left[2\tilde{a}^{ij} \tilde{a}^*_{ij} - \tilde{a}^{ij} \tilde{a}_{ij} e^{2ik_\alpha x^\alpha} - \tilde{a}^{*\, ij} \tilde{a}^*_{ij} e^{-2ik_\alpha x^\alpha} \right]. \tag{55}$$

The contribution to $t_{\mu\nu}$ coming from the combination of terms with c^σ in (49) is identically zero if $q^\alpha q_\alpha = 0$, and it is not identically zero if $q^\alpha q_\alpha \neq 0$, but in this case, it vanishes after averaging over 1 wave-period or 1 wave-length [35]. The cross-terms, containing products of $a^{\mu\nu}$ and c^α, also do not contribute to $t^{\mu\nu}$.

For a plane wave propagating in z-direction, i.e. for $k_\alpha = (k_0,\ 0,\ 0,\ k_3)$ and $k_0^2 - k_3^2 = 0$, the energy-momentum tensor (55) depends only on

$$\tilde{a}^{11} = \frac{1}{2}(a^{11} - a^{22}) = -\tilde{a}^{22} \quad \text{and} \quad \tilde{a}^{12} = a^{12}.$$

The numerical values of $a^{\mu\nu}$ are determined by the source of gravitational waves. In the presence of matter, the wave-equations are

$$h^{\mu\nu,\alpha}{}_{,\alpha} + \eta^{\mu\nu} h^{\alpha\beta}{}_{,\alpha,\beta} - h^{\nu\alpha,\mu}{}_{,\alpha} - h^{\mu\alpha,\nu}{}_{,\alpha} = 2\kappa T^{\mu\nu}. \tag{56}$$

It is assumed that the field $h^{\mu\nu}(x^\alpha)$ is Fourier-expanded:

$$h^{\mu\nu}(x^\alpha) = \frac{1}{(2\pi)^4} \int a^{\mu\nu}(k_\alpha) e^{ik_\alpha x^\alpha} d^4k \tag{57}$$

Then, the amplitudes $a^{\mu\nu}(k_\alpha)$ are determined by Fourier components of the retarded solution to Eq. (56). Assuming that the distance R_0 to the source is large, one obtains

$$a^{\mu\nu}(k_\alpha) = \frac{2\kappa}{4\pi R_0} \hat{T}^{\mu\nu}(k_\alpha), \tag{58}$$

where

$$\hat{T}^{\mu\nu}(k_\alpha) = \int \left(\int T^{\mu\nu}(t_r, r_0) d^3 r_0 \right) e^{-ik_\alpha x^\alpha} d^4x,$$

and t_r is retarded time. Therefore, we obtain

$$\tilde{a}^{11} = -\tilde{a}^{22} = \frac{1}{2} \frac{2\kappa}{4\pi R_0} [\hat{T}^{11}(k_\alpha) - \hat{T}^{22}(k_\alpha)], \quad \tilde{a}^{12} = \frac{2\kappa}{4\pi R_0} \hat{T}^{12}(k_\alpha). \tag{59}$$

The action of a gravitational wave on free test particles can be discussed in terms of the geodesic deviation equation:

$$\frac{D^2 \xi^\alpha}{ds^2} = R^\alpha{}_{\mu\nu\sigma} u^\mu u^\nu \xi^\sigma,$$

where ξ^α is the separation between world-lines of two nearby freely falling particles. Assuming that the reference particle in the origin of the coordinate system is at rest, i.e. $u^\alpha = (1,\ 0,\ 0,\ 0)$, the geodesic deviation equation reduces to

$$\frac{d^2\xi^i}{c^2dt^2} = R^i{}_{0j0}\xi^j. \tag{60}$$

The Riemann tensor is given by

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left[h_{\alpha\nu,\mu,\beta} + h_{\mu\beta,\alpha,\nu} - h_{\alpha\beta,\mu,\nu} - h_{\mu\nu,\alpha,\beta} - \frac{1}{2}\left(\eta_{\alpha\nu}h_{,\mu\beta} + \eta_{\mu\beta}h_{,\alpha,\nu} - \eta_{\alpha\beta}h_{,\mu,\nu} - \eta_{\mu\nu}h_{,\alpha,\beta}\right)\right] \tag{61}$$

Calculating (61) with the help of (49) one finds that the terms with c^μ cancel out identically. Moreover, using (51) and (52) one finds that the components $R^i{}_{0j0}$, participating in Eq. (60), depend only on $\tilde{a}^{ij}$:

$$R_{i0j0} = \frac{1}{2}k_0^2\tilde{a}_{ij}e^{ik_\alpha x^\alpha}. \tag{62}$$

Therefore, for a wave propagating in z-direction, equations (60) read:

$$\begin{aligned}
\frac{d^2\xi^1}{c^2dt^2} &= -\frac{1}{2}k_0^2\left[\frac{1}{2}(a^{11} - a^{22})\xi^1 + a^{12}\xi^2\right]e^{ik_0x^0}, \\
\frac{d^2\xi^2}{c^2dt^2} &= -\frac{1}{2}k_0^2\left[a^{12}\xi^1 - \frac{1}{2}(a^{11} - a^{22})\xi^2\right]e^{ik_0x^0}, \\
\frac{d^2\xi^3}{c^2dt^2} &= 0.
\end{aligned} \tag{63}$$

Recalling equations (59), one relates the deformation pattern of a set of test particles with the energy-momentum tensor $T^{\mu\nu}$ of the gravitational-wave source:

$$\begin{aligned}
\frac{d^2\xi^1}{c^2dt^2} &= -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right)\xi^1 + \hat{T}^{12}\xi^2\right]e^{ik_0x^0}, \\
\frac{d^2\xi^2}{c^2dt^2} &= -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\hat{T}^{12}\xi^1 - \frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right)\xi^2\right]e^{ik_0x^0}, \\
\frac{d^2\xi^3}{c^2dt^2} &= 0.
\end{aligned} \tag{64}$$

B. Free gravitational waves in the finite-range gravity.

Source-free equations are now Eqs. (46), (47) and (45). We shall consider the Fierz-Pauli case, i.e. Eq. (41), separately. Plane-wave solutions to (46) and (47) are given by

$$h = Ce^{iq_\alpha x^\alpha} + c.c., \tag{65}$$

$$H^{\mu\nu} = a^{\mu\nu}e^{ik_\alpha x^\alpha} + c.c., \tag{66}$$

where the wave-vectors q_α and k_α satisfy the conditions

$$q_\alpha q^\alpha = \beta^2, \qquad k_\alpha k^\alpha = \alpha^2.$$

As a consequence of (53), 10 members of the matrix $a^{\mu\nu}$ are restricted by 5 constraints:

$$a^{\mu\nu}\eta_{\mu\nu} = 0, \qquad a^{\mu\nu}k_\nu = 0. \tag{67}$$

Having found h and $H^{\mu\nu}$ we can write down the original quantities $h^{\mu\nu}$, using for this purpose the relationship (42). In doing that, it is convenient to introduce a new notation:

$$A = -C\frac{\alpha^2}{2(2\alpha^2+\beta^2)}.$$

Then,

$$h^{\mu\nu} = a^{\mu\nu}e^{ik_\alpha x^\alpha} - \eta^{\mu\nu}Ae^{iq_\alpha x^\alpha} + 2\frac{q^\mu q^\nu}{\alpha^2}Ae^{iq_\alpha x^\alpha} - \eta^{\mu\nu}\frac{\beta^2}{\alpha^2}Ae^{iq_\alpha x^\alpha}. \tag{68}$$

With this $h^{\mu\nu}$, Eq. (45) is satisfied automatically.

One can note that the term with $\eta^{\mu\nu}$ in Eq. (68) (and indeed the original term with $\gamma^{\mu\nu}$ in Eq. (42)) is split into two parts. This has been done on purpose. One can see that the last two terms in Eq. (68) have the structure of the c^μ-terms in Eq. (49) (with $c^\mu \propto q^\mu$). We know that these terms do not contribute to the observational effects of the geodesic deviation equation, because of cancellation of these terms in the Riemann tensor (61). This fact simplifies calculations, but it also shows that the mere appearance of α^2 in the denominator of the field variables does not represent a danger, even if α^2 is eventually sent to zero. The deviations and "discontinuities" in the observational predictions of the massive theory are not related to these "divergencies" arising from α^2 in the denominator. We shall return to this point later on.

C. Emission of gravitational waves in the finite-range gravity.

The gravitational wave amplitudes $a^{\mu\nu}, A$ are determined by the source, emitting gravitational waves. In the presence of $T^{\mu\nu}$, the linearised field equations are

$$h^{\mu\nu,\alpha}{}_{,\alpha} + \eta^{\mu\nu}h^{\alpha\beta}{}_{,\alpha,\beta} - h^{\nu\alpha,\mu}{}_{,\alpha} - h^{\mu\alpha,\nu}{}_{,\alpha} + 2\left[2k_1h^{\mu\nu} - \eta^{\mu\nu}\left(k_1+2k_2\right)h\right] = 2\kappa T^{\mu\nu}. \tag{69}$$

Repeating the same steps that has led us to Eqs. (46), (47) and (45), and taking into account the independent equation

$$T^{\mu\nu}{}_{,\nu} = 0, \tag{70}$$

we arrive at the same Eq. (45), but equations for h and $H^{\mu\nu}$ become inhomogeneous:

$$\Box h + \beta^2 h = \frac{2\alpha^2+\beta^2}{3\alpha^2}2\kappa T; \tag{71}$$

$$\Box H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 2\kappa\left(T^{\mu\nu} - \frac{1}{3\alpha^2}T^{,\mu,\nu} + \frac{1}{6\alpha^2}\eta^{\mu\nu}\Box T - \frac{1}{6}\eta^{\mu\nu}T\right), \tag{72}$$

where $T = g_{\mu\nu}T^{\mu\nu} \approx \eta_{\mu\nu}T^{\mu\nu}$.

The emitted field is determined by the retarded solutions to Eqs. (71), (72). When writing down these solutions, we closely follow the recipes and conventions of the book [36], and we refer to this book for further details. Let the distance between the point $\mathbf{r_0}$ within a compact source and the observation point $\mathbf{r}$ be $R = |\mathbf{r}-\mathbf{r_0}|$. The retarded time t_r is $t_r = t - R/c$. Let us start from Eq. (71) and its solution:

$$h(t,\mathbf{r}) = \frac{2\kappa}{4\pi}\frac{2\alpha^2+\beta^2}{3\alpha^2}\int_0^t dt_0 \int\left[\frac{1}{R}\delta(t-t_0-R/c) - \frac{\beta}{\sqrt{(t-t_0)^2-(R/c)^2}}J_1\left(\beta c\sqrt{(t-t_0)^2-(R/c)^2}\right)u(t-t_0-R/c)\right]T\, d^3r_0, \tag{73}$$

where $J_1(x)$ is a Bessel function, $u(x)$ is a step function. We assume that $r \gg r_0$, so that $R \approx R_0$, and we use the small-argument approximation for the Bessel function, $J_1(x) \approx x/2$. Then,

$$h(t,\mathbf{r}) = \frac{2\kappa}{4\pi}\frac{2\alpha^2+\beta^2}{3\alpha^2}\left[\frac{1}{R_0}\int T(t_r,r_0)d^3r_0 - \frac{\beta^2 c}{2}\int_0^{t_r}\int T(t_0,r_0)dt_0d^3r_0 + O(\beta^4)\right].$$

In a similar manner, one finds the approximate solution to Eq. (72):

$$H^{\mu\nu}(t,\mathbf{r}) = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left\{\int T^{\mu\nu}(t_r,r_0)d^3r_0 - \frac{1}{3\alpha^2}\partial^\mu\partial^\nu\int T(t_r,r_0)d^3r_0 + \frac{1}{6\alpha^2}\eta^{\mu\nu}\Box\int T(t_r,r_0)d^3r_0 - \right.$$

$$\frac{1}{6}\eta^{\mu\nu}\int T(t_r,r_0)d^3r_0 - \frac{cR_0\alpha^2}{2}\left[\int_0^{t_r}\int T^{\mu\nu}(t_0,r_0)dt_0d^3r_0 - \frac{1}{3\alpha^2}\partial^\mu\partial^\nu\int_0^{t_r}\int T(t_0,r_0)dt_0d^3r_0 + \right.$$

$$\left.\left.\frac{1}{6\alpha^2}\eta^{\mu\nu}\Box\int_0^{t_r}\int T(t_0,r_0)dt_0d^3r_0 - \frac{1}{6}\eta^{\mu\nu}\Box\int_0^{t_r}\int T(t_0,r_0)dt_0d^3r_0\right] + O(\alpha^4)\right\}.$$

The terms involving integration over t_0 are "tail effects" reflecting the fact that the d'Alembert operator in Eqs. (71), (72) is augmented by terms $\beta^2 h$ and $\alpha^2 H^{\mu\nu}$, respectively, causing dispersion of waves. Ironically, these cumbersome and difficult to calculate "tail effects" turn out to be unimportant for what follows below, but it was not easy to envisage this fact in advance.

Having found h and $H^{\mu\nu}$, we can write down the $h^{\mu\nu}$, using the relationship (42). We again assume that h and $H^{\mu\nu}$ are Fourier decomposed in expansions similar to Eq. (57). Then, we obtain for the Fourier amplitudes:

$$a^{\mu\nu}(k^\sigma) = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[\hat{T}^{\mu\nu}(k^\sigma) + \frac{k^\mu k^\nu}{3\alpha^2}\hat{T}(k^\sigma) - \frac{1}{3}\eta^{\mu\nu}\hat{T}(k^\sigma) - \frac{cR_0}{6}k^\mu k^\nu\tilde{T}(k^\sigma) + O(\alpha^2)\right],$$

$$A(k^\sigma) = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[-\frac{1}{6}\hat{T}(k^\sigma) + O(\beta^2)\right],$$

where

$$\hat{T}^{\mu\nu}(k^\sigma) = \int\left(\int T^{\mu\nu}(t_r,r_0)d^3r_0\right)e^{-ik_\sigma x^\sigma}d^4x; \quad \hat{T}(k^\sigma) = \int\left(\int T(t_r,r_0)d^3r_0\right)e^{-ik_\sigma x^\sigma}d^4x;$$

$$\tilde{T}(k^\sigma) = \int\left(\int_0^{t_r}dt_0\int T(t_0,r_0)d^3r_0\right)e^{-ik_\sigma x^\sigma}d^4x.$$

Clearly, since Eq. (70) requires

$$\hat{T}^{\mu\nu}k_\nu = 0, \tag{74}$$

the derived matrix $a^{\mu\nu}(k^\sigma)$ satisfies (in the leading order by α^2) the restrictions (67).

We now consider a wave propagating in z-direction, and we neglect in $a^{\mu\nu}$ and A the small contributions of order α^2, β^2 and higher. Then,

$$A = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[-\frac{1}{6}\hat{T}\right]; \quad a^{11} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[\hat{T}^{11} + \frac{1}{3}\hat{T}\right]; \quad a^{12} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\hat{T}^{12}; \quad a^{13} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\hat{T}^{13};$$

$$a^{22} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[\hat{T}^{22} + \frac{1}{3}\hat{T}\right]; \quad a^{23} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\hat{T}^{23}; \quad a^{33} = \frac{2\kappa}{4\pi}\frac{1}{R_0}\left[\hat{T}^{33} + \frac{k^3k^3}{3\alpha^2}\hat{T} + \frac{1}{3}\hat{T} - \frac{cR_0}{6}k^3k^3\tilde{T}\right]. \tag{75}$$

One can compare these amplitudes with those of GR, Eq. (58). We see that the wave-field produced by one and the same distribution of matter is different, depending on whether the emission is governed by equations of general relativity or equations of the finite-range gravity. The problem now is to quantify this difference in terms of observable effects, and to explore the massless limit $\alpha^2 \to 0$, $\beta^2 \to 0$ of the massive theory.

D. Observable manifestations of gravitational waves in the finite-range gravity.

The geodesic deviation equation (60) is valid regardless of whether the participating field $h^{\mu\nu}$ is a solution of equations of GR or of the finite-range theory. However, the solutions for $h^{\mu\nu}$ are different and, therefore, the explicit expressions for R_{i0j0} are not the same. Using (68) and (67), we now find

$$R_{i0j0} = \frac{1}{2}\left(k_0^2 a_{ij} - k_i k_j a^{lm}\eta_{lm} + k_i a_j^l k_l + k_j a_i^l k_l\right)e^{ik_\alpha x^\alpha} + \frac{1}{2}A\left(q_0^2\eta_{ij} + q_i q_j\right)e^{iq_\alpha x^\alpha}. \tag{76}$$

Therefore, Eq. (60) takes the form

$$\frac{d^2\xi^i}{c^2dt^2} = \frac{1}{2}\left(k_0^2 a^i_j \xi^j - k^i a^{lm}\eta_{lm}k_j\xi^j + k^i a^l_j k_l \xi^j + a^{il}k_l k_j \xi^j\right)e^{ik_0x^0} + \frac{1}{2}A\left(q_0^2\xi^i + q^i q_j \xi^j\right)e^{iq_0x^0}. \tag{77}$$

As before, we consider a wave propagating in z-direction. This means that $k_0^2 - k_3^2 = \alpha^2$, $q_0^2 - q_3^2 = \beta^2$, and a consequence of Eq. (67) reads $a^{11} + a^{22} + \frac{\alpha^2}{k_0^2}a^{33} = 0$. For concreteness, we will be interested in response of the test particles to waves of a given fixed frequency, so we put $q_0 = k_0$ in Eq. (77). Then, equations (77) take the simpler form:

$$\frac{d^2\xi^1}{c^2dt^2} = -\frac{1}{2}k_0^2\left[(a^{11}-A)\xi^1 + a^{12}\xi^2 + \frac{\alpha^2}{k_0^2}a^{13}\xi^3\right]e^{ik_0x^0},$$

$$\frac{d^2\xi^2}{c^2dt^2} = -\frac{1}{2}k_0^2\left[a^{12}\xi^1 + (a^{22}-A)\xi^2 + \frac{\alpha^2}{k_0^2}a^{23}\xi^3\right]e^{ik_0x^0},$$

$$\frac{d^2\xi^3}{c^2dt^2} = -\frac{1}{2}\alpha^2\left[a^{13}\xi^1 + a^{23}\xi^2 + \frac{\alpha^2}{k_0^2}a^{33}\xi^3\right]e^{ik_0x^0} + \frac{1}{2}\beta^2 A\xi^3 e^{ik_0x^0}.. \tag{78}$$

Clearly, the small terms proportional to α^2 and β^2 provide the extra components of motion as compared with the GR behaviour (63).

The next step is to use the calculated amplitudes (75) in Eq. (63). In doing that, we will be taking into account a consequence of Eq. (74) which reads: $\hat{T} \equiv \hat{T}^{00} - \hat{T}^{11} - \hat{T}^{22} - \hat{T}^{33} = -\hat{T}^{11} - \hat{T}^{22} - \frac{\alpha^2}{k_0^2}\hat{T}^{33}$. A little calculation shows that

$$a^{11} - A = \frac{2\kappa}{4\pi R_0}\left[\frac{1}{2}(\hat{T}^{11} - \hat{T}^{22}) - \frac{1}{2}\frac{\alpha^2}{k_0^2}\hat{T}^{33}\right],$$

$$a^{22} - A = \frac{2\kappa}{4\pi R_0}\left[-\frac{1}{2}(\hat{T}^{11} - \hat{T}^{22}) - \frac{1}{2}\frac{\alpha^2}{k_0^2}\hat{T}^{33}\right],$$

$$\frac{\alpha^2}{k_0^2}a^{33} \approx \frac{2\kappa}{4\pi R_0}\left[\frac{1}{3}\hat{T}\right].$$

Therefore, equations (78) reduce to

$$\frac{d^2\xi^1}{c^2dt^2} = -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\frac{1}{2}(\hat{T}^{11} - \hat{T}^{22})\xi^1 + \hat{T}^{12}\xi^2\right]e^{ik_0x^0} + O(\alpha^2, \beta^2),$$

$$\frac{d^2\xi^2}{c^2dt^2} = -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\hat{T}^{12}\xi^1 - \frac{1}{2}(\hat{T}^{11} - \hat{T}^{22})\xi^2\right]e^{ik_0x^0} + O(\alpha^2, \beta^2),$$

$$\frac{d^2\xi^3}{c^2dt^2} = -\frac{2\kappa}{4\pi R_0}\left[\frac{\alpha^2}{2}\left(\hat{T}^{13}\xi^1 + \hat{T}^{23}\xi^2\frac{1}{3}\hat{T}\xi^3 + O(\alpha^2)\right) + \frac{\beta^2}{2}\left(\frac{1}{6}\hat{T}\xi^3 + O(\beta^2)\right)\right]e^{ik_0x^0}. \tag{79}$$

In the massless limit, when both α^2 and β^2 tend to zero, equations (79) approach equations (64), and, hence, all the observational manifestations of the massive gravitational waves tend to those of GR. [Certain observational restrictions on gravitational waves propagating with the speed different from c have been discussed previously [37], [38].]

E. The Fierz-Pauli case.

It was shown above that the smooth transition to GR is achieved when both parameters α^2 and β^2 are sent to zero. The Fierz-Pauli coupling violates this requirement, as it postulates that $k_1 + k_2 = 0$. From the viewpoint of the 2-parameter finite-range gravity this choice of k_1 and k_2 corresponds to the limit of $\beta^2 = \infty$. In these circumstances, strong deviations from GR should be expected on the grounds of physical intuition, independently of the value of the remaining free parameter α^2. Although the deviations are expected, and even in the limit $\alpha^2 \to 0$, it is interesting to study the Fierz-Pauli theory on its own, regardless of its place in the 2-parameter family.

The starting point of the discussion is equations (69), in which k_2 is taken to be equal to $-k_1$. In particular, the source-free equations reduce to the set of equations (41). The quantities $H^{\mu\nu}$ coincide with $h^{\mu\nu}$, as is seen from Eq. (42). The wave-equations in the presence of matter sources can be derived anew, but in fact they can also be recovered from the existing equations (45), (71), (72), if one takes the limit $k_1 + k_2 = 0$ ($\beta^2 \to \infty$). In particular, Eq. (71) now reads:

$$h = \frac{1}{3\alpha^2} 2\kappa T. \tag{80}$$

The field degree of freedom represented by h ($spin - 0$ graviton) has lost the ability to be radiated away. Moreover, h vanishes everywhere outside the matter source, and h can be non-zero only within the region occupied by matter with $T \neq 0$. As for equations (72), they are exactly the same as before, but $H^{\mu\nu} \equiv h^{\mu\nu}$.

Far away from the radiating source, the gravitational wave field can still be written in the form of Eq. (68), with $a^{\mu\nu}$ satisfying the constraints (67), but the amplitude C in this equation is strictly zero. The retarded solution to Eq. (72) produces the same amplitudes $a^{\mu\nu}$ as in formula (74), but $A \equiv 0$ and $A\beta^2 \equiv 0$. The necessary change to the geodesic deviation equation (77) consists in dropping out the term with A. For a wave propagating in z-direction, Eqs. (78) retain their form, but with $A = 0$ and $A\beta^2 = 0$. Since

$$\hat{T}^{11} + \frac{1}{3}\hat{T} = \frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right) - \frac{1}{6}\hat{T} - \frac{\alpha^2}{2k_0^2}\hat{T}^{33},$$

and

$$\hat{T}^{22} + \frac{1}{3}\hat{T} = -\frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right) - \frac{1}{6}\hat{T} - \frac{\alpha^2}{2k_0^2}\hat{T}^{33},$$

equations (78), in the limit $\alpha^2 \to 0$, take the form

$$\begin{aligned}
\frac{d^2\xi^1}{c^2dt^2} &= -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right)\xi^1 - \frac{1}{6}\hat{T}\xi^1 + \hat{T}^{12}\xi^2\right]e^{ik_0x^0},\\
\frac{d^2\xi^2}{c^2dt^2} &= -\frac{1}{2}k_0^2\frac{2\kappa}{4\pi R_0}\left[\hat{T}^{12}\xi^1 - \frac{1}{2}\left(\hat{T}^{11} - \hat{T}^{22}\right)\xi^2 - \frac{1}{6}\hat{T}\xi^2\right]e^{ik_0x^0},\\
\frac{d^2\xi^3}{c^2dt^2} &= 0.
\end{aligned} \tag{81}$$

Equations (81) of the Fierz-Pauli theory should be compared with the equations (64) of GR. We see that the observational manifestations of gravitational waves differ from those in GR even in the limit of $\alpha^2 \to 0$. The deformation pattern of test particles acquires the additional "common mode" motion with the amplitude proportional to $\hat{T}/6$. This centrally-symmetric motion can be associated with the survived $helicity - 0$ polarisation state of the $spin - 2$ graviton. For typical astrophysical sources, this extra component of motion is not smaller than the GR components. This means that the gravitational wave source should be emitting, at least, a double amount of energy as compared with GR. The future gravitational wave observations will be capable of putting direct experimental limits on the presence of the "common mode" component. However, the existing observations of binary pulsars are already sufficient to reject this particular modification of GR. Indeed, it is known that the gravitational wave flux from the binary pulsar PSR 1913+16 cannot deviate from the GR prediction by a 1 percent [39], let alone to be a factor of 2 different. The important lesson to be learnt from this study is the manner in which gravitational-wave considerations constrain the possible massive theories. What is dangerous, is the potentially large difference in the radiation process itself, and not the tiny discrepancies, altogether vanishing in the limit $\alpha^2 \to 0$, in the propagation speeds of gravitational waves.

VI. BLACK HOLES

The main result of this Section is the astonishing replacement of the Schwarzschild solution by a solution without an event horizon. Below, we derive and explain this non-linear solution of the massive gravity. However, we begin with the linear approximation to the problem of static spherically-symmetric gravitational field, both, in GR and in the massive gravity. We show that the conclusions of both theories are practically identical at the intermediate distances from the central source. We later use this linearised approximation as the starting point for the numerical and analytical non-linear treatment.

At the beginning of this Section we perform calculations in Lorentzian coordinates (3), but we later use also the spatially-spherical coordinates (24). Since the field is static and spherically-symmetric, all the components of the field are functions of $r = \sqrt{x^2+y^2+z^2}$. The d'Alembert operator $\Box$ is effectively replaced by the Laplace operator $\triangle \equiv \eta^{kl}\partial_k\partial_l$, where ∂_k is the ordinary partial derivative: $\partial_k \equiv \frac{\partial}{\partial x^k}$. Since the gravitational constant G enters the equations only in the product with the mass M of the central source, we write M instead of GM, effectively putting $G = 1$. We also put $c = 1$. The products αM and βM are dimensionless.

A. Static spherically-symmetric gravitational field in linear approximation

The equations and solutions for linearised static fields are similar to the equations and solutions for weak gravitational waves. One will be able to see this similarity at every level of calculations.

First, we briefly summarise the situation in GR. For the time-independent fields, the source-free equations (48) simplify, as they do not contain the time derivatives. The general static spherically-symmetric solution to these equations is given by

$$(82)$$

$$h^{00} = \frac{b}{r} + \triangle\psi, \qquad h^{0k} = \left(\frac{a}{r}\right)^{,k}, \qquad h^{kl} = -2\partial^k\partial^l\psi + \eta^{kl}\triangle\psi, \tag{83}$$

where b, a are constants of integration and ψ is an arbitrary function of r. The integration constants are determined by the source of the field. Considering a static point-like source, that is, $T^{00} = M\delta^3(r)$, $T^{0i} = 0$, $T^{ij} = 0$, one identifies the integration constants: $b = 4M$, $a = 0$. One can now calculate the Riemann tensor (61) and write down the geodesic deviation equation (60). The result of this calculation is given by

$$\frac{d^2\xi^i}{dt^2} = -\left[\frac{M}{r}\right]^{,i}_{,j}\xi^j. \tag{84}$$

We now proceed to the source-free solutions of the massive gravity. The full set of equations is the time-independent version of equations (45), (46), (47), namely:

$$h^{\mu\nu}{}_{,\nu} - \frac{\alpha^2-\beta^2}{2(2\alpha^2+\beta^2)}h^{,\mu} = 0, \tag{85}$$

$$\triangle h + \beta^2 h = 0, \tag{86}$$

$$\triangle H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 0. \tag{87}$$

Let us start from the equation (86) for the trace h. The general solution to this equation is a linear combination of two Yukawa potentials:

$$h = b_1 Y(-\beta r) + b_2 Y(\beta r), \tag{88}$$

where

$$Y(-\beta r) \equiv \frac{e^{-\beta r}}{r}, \qquad Y(\beta r) \equiv \frac{e^{\beta r}}{r}.$$

Here and below, we impose boundary conditions which require the solutions to vanish at infinity, i.e. for $r \to \infty$. In the case of solution (88), this means that the constant b_2 must be put equal to zero, so that only the term with $Y(-\beta r)$ survives.

We now turn to Eq. (87). A tensor field is regarded spherically-symmetric if it is built from a scalar function depending only on r. The general expression for $H^{\mu\nu}$ is a combination of $Y(-\alpha r)$ and $Y(\alpha r)$, and their derivatives. We retain only terms with $Y(-\alpha r)$ in order to satisfy the boundary conditions. Then, the general and vanishing at infinity solution to Eq. (87) has the form

$$H^{\mu\nu} = A^{\mu\nu}Y(-\alpha r) + V^\mu\partial^\nu Y(-\alpha r) + V^\nu\partial^\mu Y(-\alpha r) + B\partial^\nu\partial^\mu Y(-\alpha r), \tag{89}$$

where $A^{\mu\nu}$, V^{μ}, B are constants. Since the $H^{\mu\nu}$ satisfies the conditions (43), not all constants are arbitrary; they are restricted by the relationships:

$$A^{00} = -2A, \quad A^{0k} = 0, \quad A^{kl} = A\eta^{kl}, \quad V^{\mu} = 0, \quad B = \frac{A}{\alpha^2},$$

where $A = \eta_{\mu\nu}A^{\mu\nu}$. With these constants, solution (89) takes on the form

$$H^{00} = -2AY(-\alpha r), \qquad H^{0k} = 0, \qquad H^{kl} = A\eta^{kl}Y(-\alpha r) + \frac{A}{\alpha^2}\partial^k\partial^l Y(-\alpha r). \tag{90}$$

Having found h and $H^{\mu\nu}$, we can calculate $h^{\mu\nu}$. For this purpose, we use formula (42) written as

$$h^{\mu\nu} = H^{\mu\nu} + \frac{1}{2\alpha^2+\beta^2}h^{,\mu,\nu} + \frac{\alpha^2+\beta^2}{2(2\alpha^2+\beta^2)}\eta^{\mu\nu}h.$$

Substituting solutions for h and $H^{\mu\nu}$ into this expression, and rearranging the terms, we arrive at

$$h^{00} = \left[-\frac{5}{2}AY(-\alpha r) + DY(-\beta r)\right] + \triangle\left(\frac{1}{\alpha^2}\Psi\right), \qquad h^{0k} = 0, \tag{91}$$

$$h^{kl} = -\eta^{kl}\Psi - 2\partial^k\partial^l\left(\frac{1}{\alpha^2}\Psi\right) + \eta^{kl}\triangle\left(\frac{1}{\alpha^2}\Psi\right), \tag{92}$$

where

$$\Psi = -\frac{1}{2}AY(-\alpha r) - DY(-\beta r) \qquad \text{and} \qquad D = \frac{\alpha^2}{2(2\alpha^2+\beta^2)}b_1.$$

With these $h^{\mu\nu}$, equations (85) are satisfied automatically. Thus, we are left with two arbitrary constants: b_1 and A.

The constants b_1 and A are determined by the source of the field. Clearly, the inhomogeneous equations for h and $H^{\mu\nu}$ are the same as the previously discussed gravitational wave equations (71), (72), but with the "box" being replaced by the "triangle":

$$\triangle h + \beta^2 h = \frac{2\alpha^2+\beta^2}{3\alpha^2}2\kappa T, \tag{93}$$

$$\triangle H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 2\kappa\left(T^{\mu\nu} - \frac{1}{3\alpha^2}T^{,\mu,\nu} + \frac{1}{6\alpha^2}\eta^{\mu\nu}\triangle T - \frac{1}{6}\eta^{\mu\nu}T\right). \tag{94}$$

We consider a static point-like source with $T^{\mu\nu}$ used before, namely, $T^{00} = M\delta^3(r)$, $T^{0i} = 0$, $T^{ij} = 0$. Then, solution for h satisfying the adopted boundary conditions at infinity is given by [36]:

$$h = \frac{2\kappa}{4\pi}\frac{2\alpha^2+\beta^2}{3\alpha^2}\int\frac{e^{-\beta|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}T(r_0)d^3r_0 = \frac{2(2\alpha^2+\beta^2)}{\alpha^2}\frac{2M}{3}Y(-\beta r).$$

Comparing this solution with (88), we identify the constant b_1 and, hence, the constant D:

$$D = \frac{2M}{3}.$$

In a similar manner, one finds solution to Eq. (94). However, to identify the constant A, it is sufficient to write down only one component, H^{00}:

$$H^{00} = \frac{8M}{3}Y(-\alpha r).$$

Comparing this solution with (90), we identify A:

$$A = -\frac{4M}{3}.$$

In particular, the function Ψ takes the form

$$\Psi = \frac{2M}{3}[Y(-\alpha r) - Y(-\beta r)]. \tag{95}$$

Later in this Section we will need the solution (91), (92) written in spherical coordinates (24). Applying to the tensor $h^{\mu\nu}$ a coordinate transformation from Cartesian to spherical coordinates, one finds

$$h^{00} = 4MY(-\alpha r) - \Psi + \triangle\left(\frac{1}{\alpha^2}\Psi\right), \tag{96}$$

$$h^{11} = \Psi - \frac{2}{\alpha^2}\ddot{\Psi} + \triangle\left(\frac{1}{\alpha^2}\Psi\right), \tag{97}$$

$$h^{22} = \sin^2\theta\; h^{33} = \frac{1}{r^2}\left(\Psi + \frac{1}{\alpha^2}\ddot{\Psi}\right), \tag{98}$$

where the indeces $(1, 2, 3)$ correspond to the coordinates (r, θ, ϕ), and an over-dot denotes the derivative with respect to r.

We now turn to the geodesic deviation equation in the massive gravity. We have to calculate the Riemann tensor entering Eq. (60). The last term of h^{00} in (91), and the last two terms of h^{kl} in (92), do not contribute to the Riemann tensor. This fact simplifies the calculations. Having found the contribution of the remaining terms to the Riemann tensor, we can write:

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\left[2AY(-\alpha r) + DY(-\beta r)\right]^{,i}_{\ ,j}\xi^j. \tag{99}$$

Using $A = -4M/3$, $D = 2M/3$, and expanding the Yukawa potentials in powers of small parameters $\alpha r \ll 1$, $\beta r \ll 1$, we obtain

$$\frac{d^2\xi^i}{dt^2} = -\left[\frac{M}{r} + O((\alpha r)^2, (\beta r)^2)\right]^{,i}_{\ ,j}\xi^j. \tag{100}$$

In other words, in the limit of vanishingly small parameters α^2 and β^2, we fully recover the GR result given by Eq. (84).

The situation is dramatically different in the Fierz-Pauli case. As we have already discussed above, the trace h is zero everywhere outside the source. The solution (91), (92) retain its form, but with $D = 0$. At the same time, the constant A is the same as in the general case, i.e. $A = -4M/3$. Using $D = 0$ and $A = -4M/3$ in the geodesic deviation equation (99), we arrive at

$$\frac{d^2\xi^i}{dt^2} = -\frac{4}{3}\left[MY(-\alpha r)\right]^{,i}_{\ ,j}\xi^j = -\frac{4}{3}\left[\frac{M}{r} + O(\alpha^2 r^2)\right]^{,i}_{\ ,j}\xi^j.$$

Comparing this expression with the GR result, Eq. (84), we see that there exists a finite difference, represented by the factor $4/3$, which does not vanish even in the limit of $\alpha^2 \to 0$. This difference puts the Fierz-Pauli coupling in conflict with the available static field experiments. In particular, the massless limit of the Fierz-Pauli theory contradicts the observed value of the light deflection in the gravitational field of the Sun. This conclusion has been previously reached by a number of authors [22], [7].

Finally, we have to emphasize again that the solution (91), (92) is valid at distances far away from the central source, i.e. at $r \gg M$. If, at the same time, r is much smaller than $1/\alpha$ and $1/\beta$, then r belongs to the region which we call the intermediate zone. In this zone, solution (91), (92) with $A = -4M/3$ and $D = 2M/3$, is practically indistinguishable from the linearised GR solution (82) with $b = 4M$ and $a = 0$. At very large distances, that is, in the region where αr and βr are comparable with 1 and much greater than 1, solution (91), (92) of the massive gravity strongly deviates from that of GR, as the Yukawa potentials lead to the exponential decrease of the field components $h^{\mu\nu}$ as functions of the increasing r. This is the expected behaviour, and it explains the name: the finite-range gravity. The region where r is comparable with M and smaller then M is not covered by the linear approximation. To study the behaviour of solutions in this region, we need a non-linear treatment.

B. The general non-linear equations

A static spherically-symmetric gravitational field depends on r, and therefore it is convenient to use spherical coordinates (24) and the metric tensor

$$\gamma_{00} = 1, \;\; \gamma_{11} = -1, \;\; \gamma_{22} = -r^2, \; \gamma_{33} = -r^2 \sin^2\theta. \tag{101}$$

The non-zero components of the gravitational field $h^{\mu\nu}$ can be written as

$$h^{00} = A(r), \quad h^{11} = -B(r), \quad h^{22} = -D(r), \quad h^{33} = -\frac{D(r)}{\sin^2\theta},$$

where three functions, $A(r),\ B(r),\ D(r)$, should be found from the field equations. Since we try to use as many results as possible from the text-book calculations, mostly performed in the geometrical language, we introduce three new functions, $f(r),\ f_1(r),\ R(r)$, according to the relationships

$$A = \left(\frac{R}{r}\right)^2 \sqrt{\frac{f_1}{f}} - 1, \quad B = \left(\frac{R}{r}\right)^2 \sqrt{\frac{f}{f_1}} - 1, \quad D = \frac{1}{r^2}\left(\sqrt{f f_1} - 1\right).$$

The rationale behind this notation is the following one. The tensor $g^{\mu\nu}$, calculable from Eq. (4), and the inverse tensor $g_{\mu\nu}$, calculable from $g^{\mu\nu}$, will now be described by simple and familiar expressions:

$$g^{00} = \frac{1}{f}, \quad g^{11} = -\frac{1}{f_1}, \quad g^{22} = -\frac{1}{R^2}, \quad g^{33} = -\frac{1}{R^2 \sin^2\theta}$$

and

$$g_{00} = f, \quad g_{11} = -f_1, \quad g_{22} = -R^2, \quad g_{33} = -R^2 \sin^2\theta. \tag{102}$$

Indeed, one can check these relationships by using the definition of $g_{\mu\nu}$, which follows from Eq. (4):

$$g_{00} = (1 - r^2 h^{22})\sqrt{\frac{1 - h^{11}}{1 + h^{00}}} = f(r), \quad g_{11} = -(1 - r^2 h^{22})\sqrt{\frac{1 + h^{00}}{1 - h^{11}}} = -f_1(r),$$

$$g_{22} = \frac{1}{\sin^2\theta} g_{33} = -r^2 \sqrt{(1 + h^{00})(1 - h^{11})} = -R^2(r). \tag{103}$$

Taking into account the notations (102), one can calculate the non-zero components of the Einstein tensor $G^{\mu}{}_{\nu}$:

$$G^0{}_0 = \frac{1}{f_1}\left[-2\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 + \frac{\dot{R}}{R}\frac{\dot{f}_1}{f_1} + \frac{f_1}{R^2}\right], \qquad G^1{}_1 = \frac{1}{f_1}\left[-\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{R}}{R}\frac{\dot{f}}{f} + \frac{f_1}{R^2}\right],$$

$$G^2{}_2 = G^3{}_3 = \frac{1}{2f_1}\left[-\frac{\ddot{f}}{f} - 2\frac{\ddot{R}}{R} + \frac{1}{2}\left(\frac{\dot{f}}{f}\right)^2 + \frac{1}{2}\frac{\dot{f}}{f}\frac{\dot{f}_1}{f_1} - \frac{\dot{R}}{R}\frac{\dot{f}}{f} + \frac{\dot{R}}{R}\frac{\dot{f}_1}{f_1}\right],$$

where an over-dot denotes the derivative with respect to r.

In the massive theory, we will be using the quasi-geometric equations (21). For static spherically-symmetric fields, we have only three independent equations:

$$G^0{}_0 + M^0{}_0 = 0, \tag{104}$$

$$G^1{}_1 + M^1{}_1 = 0, \tag{105}$$

$$G^2{}_2 + M^2{}_2 = 0. \tag{106}$$

The consequence of these equations, Eq. (26), can be written in the form of Eq. (27), which amounts to a single equation

$$\mathcal{M}^{rr}{}_{;r} = 0. \tag{107}$$

Before proceeding to the massive theory, it is instructive to review the derivation of the Schwarzschild solution in GR.

In GR, one puts $M^{\mu}{}_{\nu} = 0$ and solves the massless field equations $G^{\mu}{}_{\nu} = 0$. Then, equation $G^2{}_2 = 0$ is not independent. Due to Bianchi identities, this equation can be obtained as the combination of equations $G^0{}_0 = 0$ and $G^1{}_1 = 0$. (From this point of view, Eq. (107) of the massive theory is the "extra" equation, non-existent in GR.) One is left with two independent equations for three unknown functions of r: $f,\ f_1,\ R$. From equations $G^0{}_0 = 0$ and $G^1{}_1 = 0$ one finds f and f_1 in terms of R:

$$f = a\left(1 - \frac{R_g}{R}\right), \quad f_1 = \dot{R}^2\left(1 - \frac{R_g}{R}\right)^{-1}, \tag{108}$$

where a and R_g are constants of integration, while R remains an arbitrary function of r. The effective line-element takes the form

$$\mathrm{d}s^2 = a\left(1 - \frac{R_g}{R}\right)\mathrm{d}t^2 - \frac{\dot{R}^2}{\left(1 - \frac{R_g}{R}\right)}\mathrm{d}r^2 - R^2\mathrm{d}\Omega^2 = \left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 - \frac{1}{\left(1 - \frac{2M}{R}\right)}\mathrm{d}R^2 - R^2\mathrm{d}\Omega^2, \tag{109}$$

which is the familiar Schwarzschild solution. The function $R(r)$ has been announced an independent coordinate variable R, whereas the constants $a = 1$ and $R_g = 2M$ have been found from comparison of (109) with the Newtonian gravity at $R \to \infty$.

In contrast to GR, in the massive theory, there is no functions of r left arbitrary. All three functions of r: $f,\ f_1,\ R$, are determined by three independent equations: (104), (105), (106). The mass contributions $M^{\mu}{}_{\nu}$ are supposed to be calculated in terms of the functions $f,\ f_1,\ R$ and the metric tensor (101). In order to facilitate the comparison of the finite-range solution with the Schwarzschild solution, it is convenient to re-define the field variables and the metric tensor. First, we invert the function $R = R(r)$ to $r = r(R)$ and denote $r(R) \equiv X(R)$. Second, we introduce $F(R)$ according to the definition

$$F \equiv \frac{(\dot{R})^2}{f_1} = \frac{1}{X'^2 f_1},$$

where a prime denotes the derivative with respect to R: $X' = dX/dR$. Then, the tensor $g_{\mu\nu}$ reads

$$g_{00} = f, \qquad g_{11} = -\frac{1}{X'^2 F}, \qquad g_{22} = -R^2, \qquad g_{33} = -R^2 sin^2\theta\ , \tag{110}$$

so that the effective line-element

$$\mathrm{d}s^2 = f(R)\mathrm{d}t^2 - \frac{1}{F(R)}\mathrm{d}R^2 - R^2\mathrm{d}\Omega^2$$

takes the general structure of the Schwarzschild line-element (109). In GR, the functions $f(R),\ F(R)$ are given by

$$f(R) = F(R) = 1 - \frac{2M}{R}, \tag{111}$$

whereas they are expected to be given by some other formulas in the finite-range gravity. As for the metric components (101), they transform into functions of R:

$$\mathrm{d}\sigma^2 = \mathrm{d}t^2 - X'^2\mathrm{d}R^2 - X^2\mathrm{d}\Omega^2. \tag{112}$$

The field equations, including the mass contributions, will now be built from the quantities entering Eqs. (110), (112). From the field-theoretical viewpoint, we have simply performed a coordinate transformation $r = r(R)$ of the radial coordinate, and have applied this coordinate transformation to the metric tensor and to the gravitational field components. Obviously, the field equations derived from Eqs. (101), (102) in terms of $f(r),\ f_1(r),\ R(r)$, and the field equations derived from Eqs. (112), (110) in terms of $f(R),\ F(R),\ X(R)$, are exactly the same equations, if one takes into account the corresponding change of notations.

We will now write down explicitely the exact non-linear equations. In doing that, we use the notation ζ for the mass ratio:

$$\zeta = \frac{\beta^2}{\alpha^2} = \frac{{m_0}^2}{{m_2}^2}. \tag{113}$$

The first two equations (104), (105) take the form

$$-F\left[\frac{1}{R^2}+\frac{F'}{F}\frac{1}{R}\right]+\frac{1}{R^2}=-M^0_{\ 0}, \tag{114}$$

$$-F\left[\frac{1}{R^2}+\frac{f'}{f}\frac{1}{R}\right]+\frac{1}{R^2}=-M^1_{\ 1}, \tag{115}$$

where

$$M^0_{\ 0}=\frac{\alpha^2}{2(\zeta+2)}\sqrt{X'^2\frac{F}{f}}\left[\frac{3}{4}\left(\frac{R}{X}\right)^2\left(\frac{1}{X'^2Ff}-X'^2Ff\right)-(1-\zeta)\left(\frac{X}{R}\right)^2\frac{f}{X'^2F}+(2\zeta+1)f\right]$$
$$-\frac{3\beta^2}{2(\zeta+2)}\left[\frac{1}{2}\left(X'^2F-\frac{1}{f}\right)+\left(\frac{X}{R}\right)^2\right],$$

$$M^1_{\ 1}=\frac{\alpha^2}{2(\zeta+2)}\sqrt{X'^2\frac{F}{f}}\left[-\frac{3}{4}\left(\frac{R}{X}\right)^2\left(\frac{1}{X'^2Ff}-X'^2Ff\right)-(1-\zeta)\left(\frac{X}{R}\right)^2\frac{f}{X'^2F}+\right.$$
$$\left.(2\zeta+1)\frac{1}{X'^2F}\right]+\frac{3\beta^2}{2(\zeta+2)}\left[\frac{1}{2}\left(X'^2F-\frac{1}{f}\right)-\left(\frac{X}{R}\right)^2\right].$$

Obviously, in the massless GR, that is, for $M^0_{\ 0}=M^1_{\ 1}=0$, the exact solution to these equations is the familiar formula (111).

We now turn to the third equation (106). It proves more illuminating to use Eq. (107) instead of Eq. (106). The reason being that Eq. (107) gives directly the “extra” equation, which is absent in GR. Explicitely, Eq. (107) has the form

$$\frac{\alpha^2}{2\zeta+4}\frac{1}{X'}\left(\frac{R}{X}\right)^2\left(-2\frac{X''}{X'}c_1+\frac{X'}{X}c_2+\frac{f'}{f}c_0-\frac{F'}{F}c_1+\frac{c_R}{R}\right)=0, \tag{116}$$

where

$$c_R=3\left(\frac{R}{X}\right)^2\left[-3X'^2Ff+\frac{1}{X'^2Ff}\right]+2(2\zeta+1)\left[\left(\frac{R}{X}\right)^2-\frac{1}{X'^2F}+f\right]-12\zeta\sqrt{X'^2Ff},$$

$$c_0=-\frac{3}{4}\left(\frac{R}{X}\right)^2\left[3X'^2Ff+\frac{1}{X'^2Ff}\right]+(1-\zeta)\left(\frac{X}{R}\right)^2\frac{f}{X'^2F}+(2\zeta+1)f-3\zeta\sqrt{X'^2Ff},$$

$$c_1=\frac{3}{4}\left(\frac{R}{X}\right)^2\left[3X'^2Ff+\frac{1}{X'^2Ff}\right]+(1-\zeta)\left(\frac{X}{R}\right)^2\frac{f}{X'^2F}-(2\zeta+1)\frac{1}{X'^2F}+3\zeta\sqrt{X'^2Ff},$$

$$c_2=3\left(\frac{R}{X}\right)^2\left[X'^2Ff-\frac{1}{X'^2Ff}\right]+4(1-\zeta)\left(\frac{X}{R}\right)^2\frac{f}{X'^2F}+12\zeta\left(\frac{X}{R}\right)^2\sqrt{\frac{f}{X'^2F}}.$$

Since we assume that $\alpha\neq 0$, the common factor in Eq. (116) can be ignored, so that Eq. (116) simplifies to

$$-2\frac{X''}{X'}c_1+\frac{X'}{X}c_2+\frac{f'}{f}c_0-\frac{F'}{F}c_1+\frac{c_R}{R}=0. \tag{117}$$

To double-check our analytical calculations, we have verified that a direct consequence of equations (104), (105), (106), is indeed Eq. (117), as it should be. Thus, our final goal is to find three functions $f(R)$, $F(R)$, $X(R)$ from three equations (114), (115), (117).

The parameters α and β enter Eq. (117) only through the ratio (113). One convenient choice of ζ is $\zeta=1$, i.e. $\beta^2=\alpha^2$ and, equivalently, ${m_0}^2={m_2}^2$. This choice of parameters has attracted some interest in the literature, because, in this case, Eqs. (85), which constitute the linear version of the “extra” equations (27), take the form of

$h^{\mu\nu}{}_{,\nu} = 0$. In terms of $g^{\mu\nu}$, these last equations read $\left(\sqrt{-g}g^{\mu\nu}\right)_{,\nu} = 0$. In GR, these equations define the set of harmonic coordinate systems, so successfully used by Fock [2]. Our attention to this choice of ζ is guided simply by a technical simplification of exact equations that we want to solve. It is clear from the structure of equations, and some specific analytical and numerical evaluations performed for the cases $\zeta \neq 1$, that the choice of $\zeta = 1$ does not incur any loss of generality to our conclusions. The most of our analytical and numerical calculations will deal with the case $\zeta = 1$. (The linear massive theory with $\zeta = 1$, and some non-linear theories with "subsidiary" conditions, have been considered in a number of papers [21], [40], [27], [41].) When performing numerical calculations, we have used the D02CBF–NAG Fortran Library Routine which integrates ordinary differential equations from R_{in} to R_{end} using a variable-step Adams method.

C. Weak-field approximation in the case $\zeta = 1$

It is convenient to begin with the intermediate distances from the central source, where the behaviour of the sought-for solution is known from the linear theory. When $\alpha = \beta$, the function Ψ, Eq. (95), vanishes and Eqs. (96), (97), (98) simplify to

$$h^{00} = \frac{4M}{r} + O(M\alpha), \quad h^{11} = 0, \quad h^{22} = \sin^2\theta h^{33} = 0.$$

Using Eqs. (103), we can find, first, the functions $f(r),\ f_1(r),\ R(r)$:

$$f \approx 1 - \frac{2M}{r}, \quad f_1 \approx 1 + \frac{2M}{r}, \quad R \approx r + M,$$

and, then, the functions $f(R),\ F(R),\ X(R)$:

$$F \approx f \approx 1 - \frac{2M}{R}, \quad X \approx R - M \tag{118}$$

Thus, in the intermediate region, i.e. for R satisfying the inequalities

$$1 \ll \frac{R}{M} \ll \frac{1}{\alpha M}, \tag{119}$$

the behaviour of our non-linear solution is given by Eq. (118). Certainly, the exact solution of GR, Eq. (111), subject to the transformation to harmonic coordinates $X = R - M$, is also described by formulas (118), but with the symbols of approximate equality being replaced by the symbols of equality. In harmonic coordinates, the Schwarzschild solution takes the form:

$$\mathrm{d}s^2 = \frac{X-M}{X+M}\mathrm{d}t^2 - \frac{X+M}{X-M}\mathrm{d}X^2 - (X+M)^2\mathrm{d}\Omega^2.$$

The approximate solution (118) helps us to formulate the initial conditions for numerical integration of equations (114), (115), (117). As mentioned above, we reduce these equations to the case $\zeta = 1$. For purely technical reasons of computational time and resolution, we begin with an unrealistically large value of the dimensionless parameter αM: $\alpha M = \sqrt{2} \times 10^{-6}$. Later on we will discuss the variations of this parameter. The starting point of integration is $R_{in} = 5 \times 10^3 M$, so that the inequality (119) is satisfied there pretty well. The initial values of the participating functions at R_{in} are given by

$$F = f = 1 - \frac{2M}{R_{in}}, \quad X = R_{in} - M, \quad X' = 1. \tag{120}$$

The initial value of X' needs to be specified as well, since the equation for $X(R)$ (117) is a second-order differential equation.

As expected, at intermediate distances, the solution of the finite-range gravity is practically indistinguishable from the Schwarzschild solution. In Fig. 1, we show the values of f and F numerically calculated at discrete values of R (R is given in units of M). For comparison, the dashed line shows the Schwarzschild functions $f = F = 1 - 2M/R$.

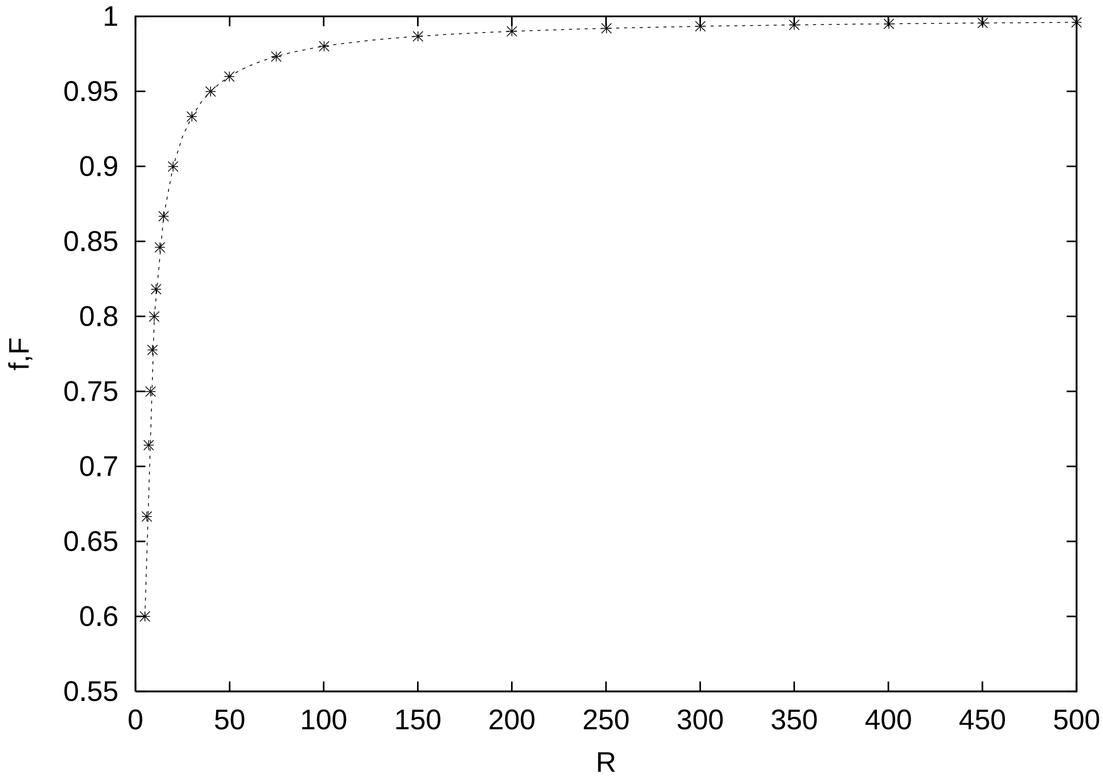

FIG. 1. Numerical solutions in the region $5M \leq R \leq 500M$. The dashed line is the Schwarzschild solution. The values of f and F in the massive gravity are shown, respectively, by $+$ and and $\times$ marks, which almost superimpose on each other.

The equally good agreement takes place between the numerically calculated function $X(R)$ of the massive gravity and the function $X = R - M$, which is a solution of the harmonic-coordinate conditions of GR. One can see in Fig. 2 that the function $X(R)$ (in units of M) is indistinguishable from a straight line $X = R - M$ for all covered values of R.

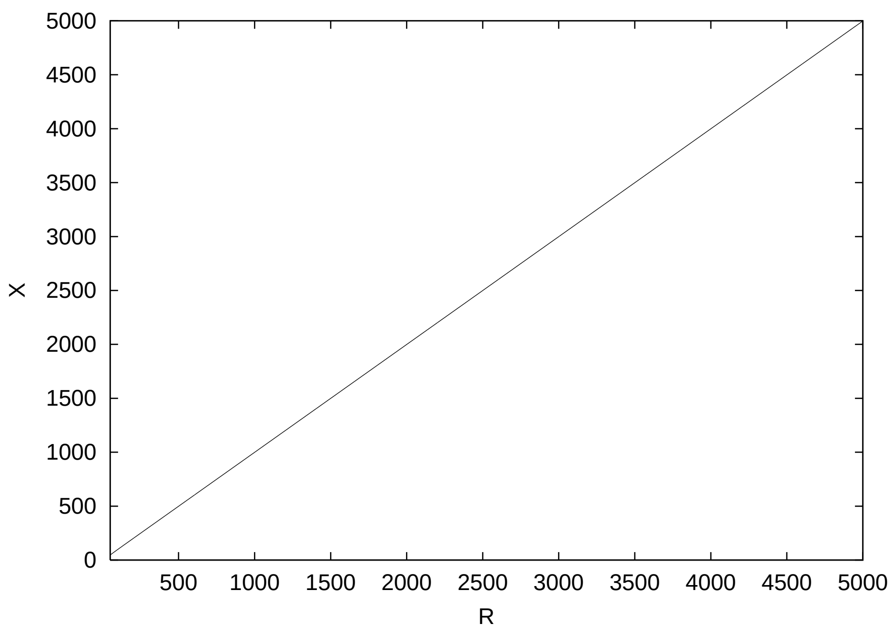

FIG. 2. Numerical solution for the function $X(R)$ in the region $50M \leq R \leq 5 \times 10^3 M$.

Certainly, the displayed numerical graphs are in agreement with analytical calculations. In the intermediate zone, equation (117) is well approximated by its linear part:

$$-2\frac{X''}{X'} + 4\frac{X}{X'}\frac{1}{R^2F} - \frac{f'}{f} - \frac{F'}{F} - \frac{4}{R} = 0. \tag{121}$$

[It is not easy to recognise equation (121) as a linear part of Eq. (117), but a straightforward way to verify Eq. (121) is to use the fact that this equation is Eq. (45): $h^{\mu\nu}{}_{;\nu} = 0$.] In the intermediate zone, equations (114), (115) can also be approximated by simpler equations, as the terms M^0_0, M^1_1 can be neglected. This leads to the approximate solution of these equations: $f = F = 1 - 2M/R$. Using these expressions for f and F in Eq. (121), one obtains a second-order differential equation for $X(R)$:

$$X^{''} + \frac{2X^{'}(R-M)}{R(R-2M)} - \frac{2X}{R(R-2M)} = 0.$$

The general solution to this equation is given by

$$X(R) = a_1(R-M) + a_2\left[\frac{R-M}{2}\ln|1-\frac{2M}{R}| + M\right], \tag{122}$$

where a_1 and a_2 are arbitrary constants. Solution (122) was first derived by Fock [2] in his study of harmonic coordinates for the Schwarzschild metric. With this approximate solution for f, F, X, one can verify that the neglected terms in equations (114), (115), (117) are indeed smaller than the retained ones.

In general relativity, it is sufficient to use only one branch of the solution (122), choosing $a_1 = 1$, $a_2 = 0$ [2]. Our initial conditions (120) do also imply $a_1 = 1$, $a_2 = 0$ in Eq. (122). This is why our numerical solution of exact equations (114), (115), (117) is practically indistinguishable, everywhere in the intermediate zone, from the Fock's exact solution: $f = F = 1 - 2M/R$, $X = R - M$. However, in the massive gravity, formula (122) is only an approximate solution. The second branch of this formula, with the logarithmically divergent term, suggests that the function $X(R)$ may start deviating from the straight line at some sufficiently small R. This is indeed the case. In Fig. 3 we show the continuation of the numerical graph for $X(R)$ from the region covered in Fig. 2 to the region $5M \le R \le 50M$.

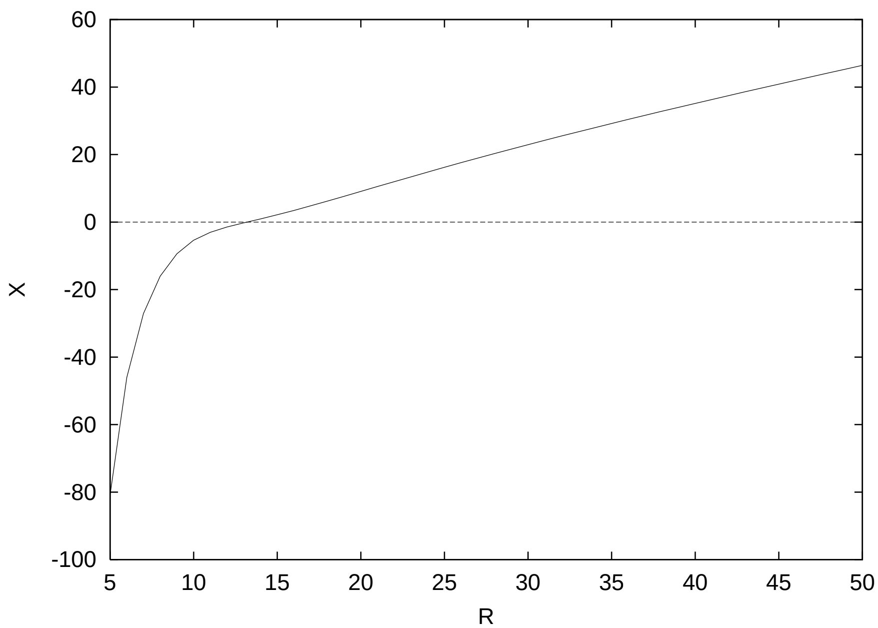

FIG. 3. Continuation of the numerical solution for $X(R)$ to the region $5M \le R \le 50M$.

It is seen from this graph that the function $X(R)$ crosses zero at some point near $R = 13M$, and then sharply drops to large negative values. For comparison, one should recall that the GR function $X = R - M$ would have crossed zero only at $R = M$. The sharp decrease of $X(R)$ continues at smaller R. This behaviour of $X(R)$ feeds back to the equations (114), (115) and drastically changes the functions $f(R)$ and $F(R)$. Clearly, this strong deviation from GR takes place at values of R approaching $2M$, i.e. in the region where the intermediate zone approximation ceases to be valid.

D. No black holes in massive gravity

In Fig. 4 we show the continuation of $X(R)$ to the interval $2.0001M \leq R \leq 2.1M$.

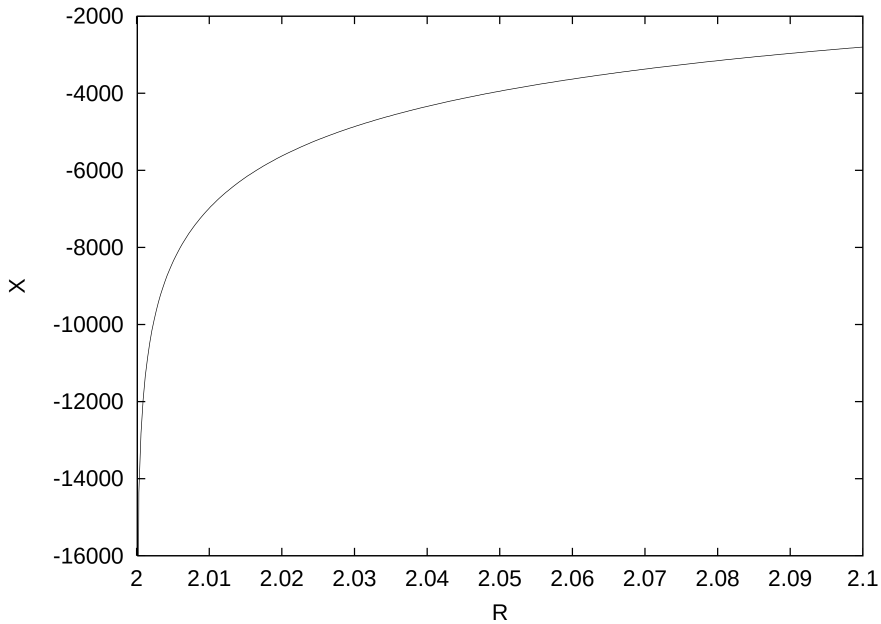

FIG. 4. Numerical function X(R) in the region $2.0001M \leq R \leq 2.1M$.

Given the initial conditions (120), the GR function $X = R - M$ would be positive and very close to M in the interval of R covered by Fig. 4. But in the massive theory, $X(R)$ is negative and continues to sharply decline below the level of $-1.4 \times 10^4 M$. Since equations (114), (115), (117) are coupled differential equations, it is natural to expect that a strong deviation from GR of one of the functions will be accompanied by strong deviations of other functions. Indeed, in Fig. 5 we show the continuation of numerical graphs for $f(R)$ and $F(R)$ to still smaller R, including the point $R = 2M$.

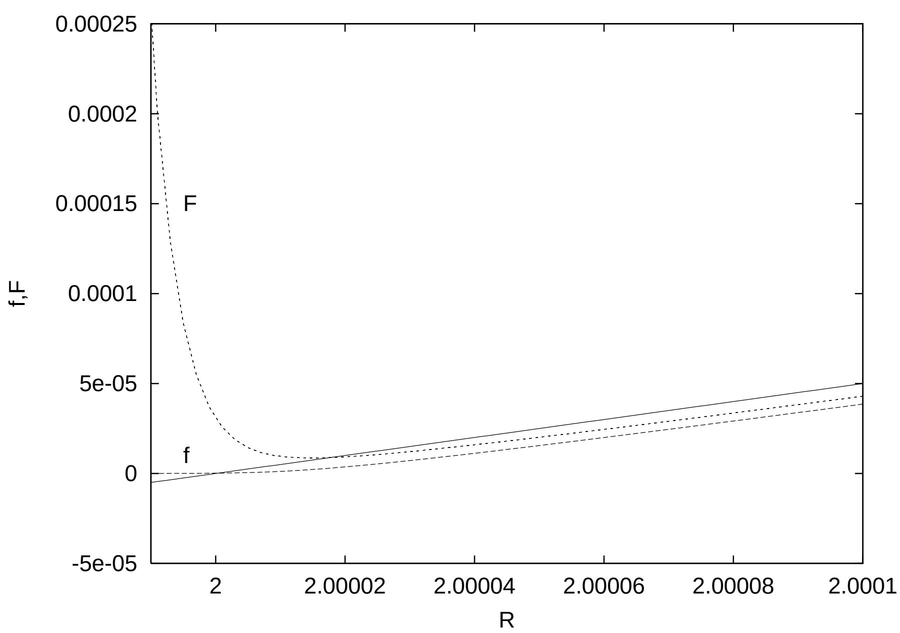

FIG. 5. The solid line is the Schwarzschild solution $f = F = 1 - 2M/R$. The dashed line is the numerical solution for $f(R)$, and the dotted line is the numerical solution for $F(R)$.

It is seen from this graph that on the way to the point $R = 2M$ the function $F(R)$ reaches a minimum, and then starts increasing again. The function $f(R)$ does not cross zero, and, presumably, approaches zero asymptotically, i.e. for $R \to 0$. The continuation of $X(R)$ to the vicinity of $R = 2M$ is shown in Fig. 6.

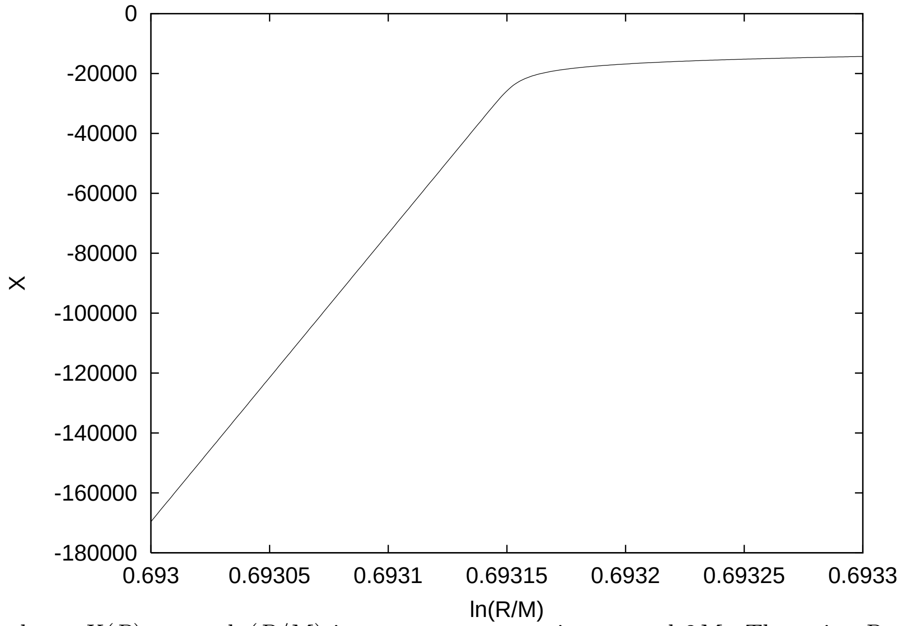

FIG. 6. The graph shows $X(R)$ versus $\ln(R/M)$ in a very narrow region around $2M$. The point $R = 2M$ corresponds to $ln(R/M) = 0.693147$.

It is difficult to analyse analytically the immediate vicinity of the point $R = 2M$, but the asymptotic analytical description is possible for much smaller R, i. e. $R \ll 2M$. This description can then be extrapolated to $R = 2M$, in pretty good agreement with the numerical analysis. We will now give this analytical description and will compare it with numerical calculations. It is likely, and we will justify this later, that the function $X(R)$ has the general form

$$X(R) = aM\, ln\left(\frac{R}{2M}\right) - bM \tag{123}$$

at $R \ll 2M$, where a and b are some constants. This behaviour is suggested by formula (122) in the limit $R \ll 2M$. If so, one can use Eq. (123) for evaluation of the leading terms in M^0_0 and M^1_1 in the limit of small R. This evaluation shows that

$$M^0_0 \approx -\frac{1}{4}\alpha^2 X'^2 F, \qquad M^1_1 \approx \frac{1}{4}\alpha^2 X'^2 F.$$

Then, equations (114), (115) take the form:

$$-F\left[\frac{1}{R^2} + \frac{F'}{F}\frac{1}{R}\right] + \frac{1}{R^2} = \nu\frac{F}{R^2},$$

$$-F\left[\frac{1}{R^2} + \frac{f'}{f}\frac{1}{R}\right] + \frac{1}{R^2} = -\nu\frac{F}{R^2},$$

where

$$\nu = \frac{1}{4}a^2(\alpha M)^2. \tag{124}$$

One can now find the exact solution to these approximate equations. It is given by

$$F(R) = C_F\left(\frac{R}{2M}\right)^{-1-\nu} + \frac{1}{1+\nu}, \tag{125}$$

$$f(R) = C_f\left(\frac{R}{2M}\right)^{-1+\nu} + \frac{C_f}{(1+\nu)C_F}\left(\frac{R}{2M}\right)^{2\nu}, \tag{126}$$

where C_F and C_f are arbitrary constants. This solution allows one to identify the leading terms in the equation (117) for X. Specifically, the main contributions to Eq. (117) are provided by the last terms in the expressions for $c_R,\ c_0,\ c_1$. The term with c_2 is subdominant. The leading terms in Eq. (117) combine to produce the approximate equation

$$X'' + \frac{1}{R}X' = 0.$$

We see that expression (123) is indeed a general solution to this equation. Having found the functions $F(R),\ f(R),\ X(R)$, one can now check that the neglected terms in all three equations (114), (115), (117) are indeed small in comparison with the retained ones. The approximate solution (125), (126), (123) is asymptotically exact in the limit $R \to 0$, i.e. in the vicinity of singularity, which we will discuss later. The arbitrary constants $C_F,\ C_f,\ a,\ b$ can only be found from comparison of this analytical solution with numerical calculations.

In Figs. 7, 8, we show the continuation of the numerically calculated functions $X(R), F(R), f(R)$ to the values of R somewhat smaller than $2M$. Since the functions change very rapidly, we switch the display from $F(R), f(R)$ to their logarithms. These graphs can be approximated by the analytical formulas (125), (126), (123), which allow us to evaluate the constants $C_F,\ C_f,\ a,\ b$ at the covered interval of R.

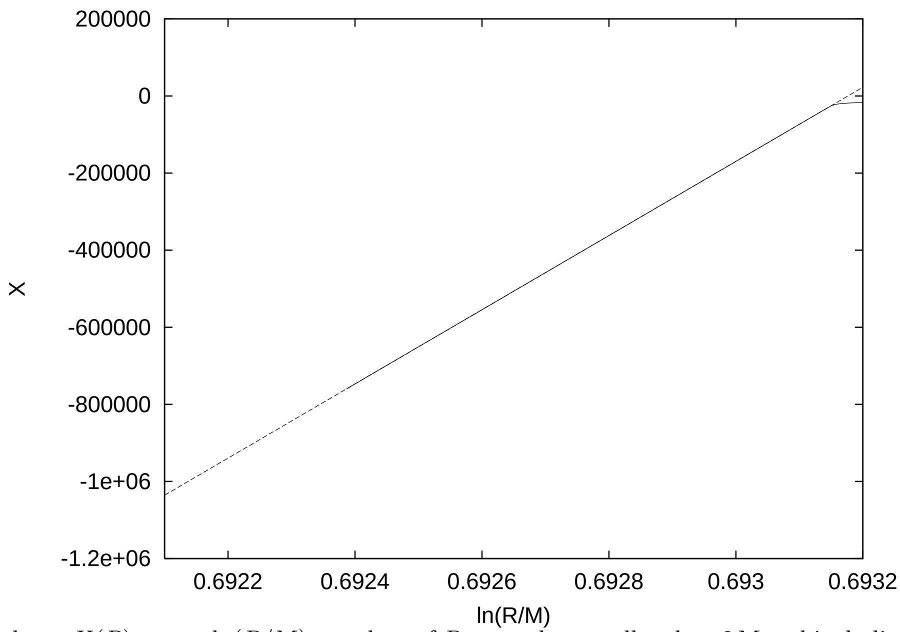

FIG. 7. The graph shows $X(R)$ versus $\ln(R/M)$ at values of R somewhat smaller than $2M$ and including $R = 2M$. The solid line is the numerical solution, while the dashed line is its analytical approximation (123) with $a = 9.62265 \times 10^8$, $b = 2.8028 \times 10^4$.

The found constant a determines the parameter ν, Eq. (124). Since $\alpha M = \sqrt{2} \times 10^{-6}$, the numerical value of ν is $\nu = 4.62977 \times 10^5$, so that one can neglect 1 in comparison with ν in Eq. (125), (126). Clearly, the deviations of the functions $f(R),\ F(R)$ from their behaviour in GR are caused by the single dimensionless parameter: αM, which was chosen to be $\alpha M = \sqrt{2} \times 10^{-6}$. In particular, the numerical values of $f(R),\ F(R),\ X(R),\ X^{'}(R)$ at $R = 2M$ are roughly expressible as various simple powers of the number $\alpha M = \sqrt{2} \times 10^{-6}$. We have varied this parameter and have checked numerically that the general behaviour of solutions remains the same, but the significant deviations of $f(R),\ F(R)$ from their GR behaviour begin closer and closer to $R = 2M$ for smaller and smaller αM. So, the field configuration resembles a black hole, in astrophysical sense, better and better, when αM decreases. We have also checked that the qualitative behaviour of numerical solutions remains the same for some other choices of ζ: $\zeta = 2$ and $\zeta = 3$.

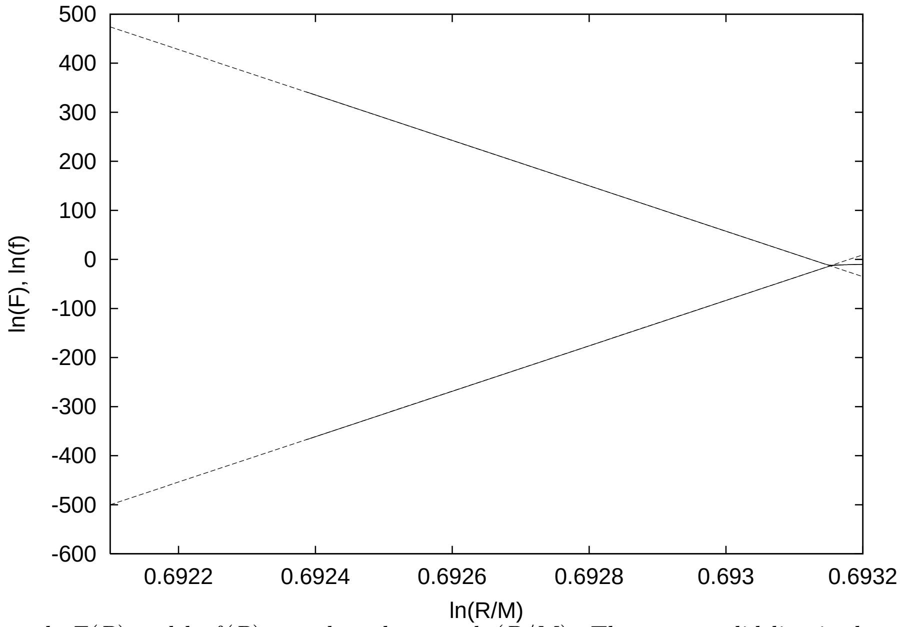

FIG. 8. The functions $\ln F(R)$ and $\ln f(R)$ are plotted versus $\ln(R/M)$. The upper solid line is the numerical solution for $F(R)$, while the dotted line is its analytical approximation (125) with $C_F = 9.8$. The lower solid line is the numerical solution for $f(R)$, while the dashed line is its analytical approximation (126) with $C_f = 1.06$.

The most interesting conclusion of this investigation is that the functions $f(R)$, $F(R)$ remain regular and positive all the way down to $R = 0$. For static spherically-symmetric metrics (102), the location of the (globally defined) event horizon is determined [42] by the condition

$$f(R) = 0.$$

In the massive gravity, the function $f(R)$ reaches zero only in the center $R = 0$, where, as we will show shortly, the physical singularity occurs. The fact that the function $f(R)$ does not vanish at any $R > 0$ shows that the static spherically-symmetric solutions of the massive gravity do not possess a regular event horizon. It is truly surprising that it requires so little to get rid of the black hole event horizon - just the inclusion of arbitrarily small mass-terms (9) in the highly non-linear gravitational equations. [3]

The Riemann invariant $I = g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} g^{\phi\lambda} R^{\mu}_{\alpha\rho\phi} R^{\nu}_{\beta\sigma\lambda}$ diverges at $R = 0$ both in GR and in the finite-range gravity. In GR, one uses the Schwarzschild solution (111) to calculate I: $I = 48M^2/R^6$. In the finite-range gravity, one uses the asymptotic formulas (125), (126) in the limit of $R \to 0$. The result of this calculation gives

$$I \approx \frac{16M^2}{R^6}(2\nu^2 - 2\nu + 3)C_F^2 \left(\frac{R}{2M}\right)^{-2\nu}.$$

The singularity of the massive gravity at $R = 0$ is a reflection of the assumed point-like nature of the source of the field and the source-free form of equations everywhere outside of the source. One can expect that for realistic extended sources, the singularity will be replaced by a very compact distribution of matter.

VII. COSMOLOGY

The cosmological solutions are based on smoothly distributed matter, so we shall work with the full set of (quasi-geometric) equations with matter sources (33):

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[3]Many would say that if a black hole (in its strict, mathematical sense) is the answer to a physical question, it must have been a very strange question. For example, the idea of unlimited collapse to a black hole seemed (apparently) so disgusting to Landau that in order to avoid this conclusion he was ready to sacrifice the laws of quantum statistics [43]. But it seems surprising to us that the disappearance of the event horizon may be caused by arbitrarily small mass-terms.

$$G^{\mu}{}_{\nu} + M^{\mu}{}_{\nu} = \kappa T^{\mu}{}_{\nu}. \tag{127}$$

The conservation equations (34) are satisfied independently of (127), and therefore the consequences of Eq. (127) are given by Eq. (26) or, equivalently, by Eq. (27).

We will be using the Lorentzian coordinates (3), and we will be interested in simplest homogeneous isotropic cosmological models. This means that the gravitational field components $h^{\mu\nu}$ depend only on time t and have a diagonal form:

$$h^{00} = A(t), \quad h^{11} = h^{22} = h^{33} = -B(t).$$

Since we want to use as many text-book calculations as possible, we introduce new field variables $a(t)$ and $b(t)$ according to the definitions

$$A = \frac{a^3}{b} - 1, \qquad B = ab - 1$$

Then, the tensor $g^{\mu\nu}$, calculable from Eq. (4), has the following non-zero components

$$g^{00} = \frac{1}{b^2}, \quad g^{11} = g^{22} = g^{33} = -\frac{1}{a^2},$$

and the inverse tensor is

$$g_{00} = b^2, \quad g_{11} = g_{22} = g_{33} = -a^2.$$

The effective line-element acquires a familiar form

$$\mathrm{d}s^2 = b^2(t)c^2\mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{128}$$

The Einstein tensor $G^{\mu}{}_{\nu}$ calculated from the effective metric (128) has the following non-zero components:

$$G^{0}{}_{0} = \frac{3}{b^2}\left(\frac{a'}{a}\right)^2, \tag{129}$$

$$G^{1}{}_{1} = G^{2}{}_{2} = G^{3}{}_{3} = \frac{1}{b^2}\left[2\left(\frac{a'}{a}\right)' + 3\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\frac{b'}{b}\right], \tag{130}$$

where a prime denotes the derivative with respect to ct: $' = \mathrm{d}/c\mathrm{d}t$.

As for the matter sources, we adopt a perfect fluid model with the (geometrical) energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu}.$$

Since $u^i = 0$ and $u^0 = 1$, the non-zero components of T^{μ}_{ν} are $T^0_0 = \varepsilon(t)$ and $T^1_1 = T^2_2 = T^3_3 = -p(t)$. The conservation equations (34) reduce to a single equation

$$\varepsilon' + 3\frac{a'}{a}(p + \varepsilon) = 0. \tag{131}$$

As the final simplification, we assume that the fluid is described by the equation of state $p(t) = q\varepsilon(t)$, where q is a constant. (For our purposes it will be sufficient to consider $-1 < q < 1$.) Then equation (131) can be integrated to produce

$$\varepsilon(t) = \frac{\varepsilon_0}{a^{3(q+1)}}, \tag{132}$$

where ε_0 is an arbitrary constant with the dimensionality of energy density. More realistic models of matter assume piece-wise equations of state, whereby the constant q is different at different intervals of cosmological evolution.

A. Homogeneous isotropic solutions in GR

It is instructive to start from the simplest Friedmann solutions of GR. The two independent Einstein equations are

$$\frac{3}{b^2}\left(\frac{a'}{a}\right)^2 = \kappa\varepsilon, \tag{133}$$

$$\frac{1}{b^2}\left[2\left(\frac{a'}{a}\right)' + 3\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\frac{b'}{b}\right] = -\kappa q\varepsilon.$$

The second equation is satisfied identically, if the first equation and Eq. (132) are satisfied. Using the relationship (132) in Eq. (133), and introducing the independent variable τ according to the definition

$$\mathrm{d}\tau = b(t)\mathrm{d}t,$$

equation (133) can be integrated to yield

$$a(\tau) = \left(\frac{\tau}{\tau_1}\right)^{\frac{2}{3(q+1)}}, \tag{134}$$

where

$$c\tau_1 = \frac{2}{\sqrt{3}(q+1)}l_0 \quad \text{and} \quad l_0 = \frac{1}{\sqrt{\kappa\varepsilon_0}}.$$

Returning back to Eq. (132) with $a(\tau)$ from Eq. (134), one finds

$$\kappa\varepsilon = \frac{4}{3(q+1)^2c^2\tau^2}. \tag{135}$$

Thus, in GR, the function $b(t)$ remains arbitrary. The effective line-element (128) can be written in the form

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 - a^2(\tau)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{136}$$

where $a(\tau)$ is called the scale factor. The independent variable τ is the absolute time elapsed since the singularity at $\tau = 0$. The values of measurable quantities, i.e. the matter energy density $\varepsilon(\tau)$, Eq. (135), and the Hubble radius

$$l_H(\tau) = c/H(\tau) = \frac{3(q+1)}{2}c\tau,$$

are completely determined by the value of the absolute time τ. The constant τ_1 (or, for this matter, the constant l_0/c) has the dimensionality of [*time*]. The constant τ_1 marks the moment of time τ when the scale factor $a(\tau)$, Eq. (134), reaches $a = 1$, and the energy density $\varepsilon(\tau)$ reaches ε_0, but the numerical value of $a(\tau)$ has no physical significance. At any chosen moment of time τ, by adjusting the constant τ_1, one can make $a > 1$ or $a < 1$, while solutions with differing constants τ_1 are observationally indistinguishable. As we shall see below, this situation changes in the massive gravity.

B. Exact cosmological equations in the finite-range gravity

The massive contributions M^0_0 and $M^1_1 = M^2_2 = M^3_3$ are directly calculable from their definitions (23). With the massive terms taken into account, the two independent field equations (127) read:

$$3\left(\frac{\dot a}{a}\right)^2 + \frac{3}{8}\frac{\alpha^2}{\zeta+2}\left[\frac{a^3}{b^3} - \frac{b}{a} + 2\zeta\left(\frac{1}{b^2} + 2\frac{b}{a} - \frac{3}{a^2}\right)\right] = \kappa\varepsilon, \tag{137}$$

$$2\left(\frac{\dot a}{a}\right)^{\cdot} + 3\left(\frac{\dot a}{a}\right)^2 - \frac{1}{8}\frac{\alpha^2}{\zeta+2}\left[3\frac{a^3}{b^3} + \frac{b}{a} - 4\frac{a}{b} + 2\zeta\left(\frac{3}{b^2} - 2\frac{b}{a} - 4\frac{a}{b} + \frac{3}{a^2}\right)\right] = -\kappa q\varepsilon, \tag{138}$$

where an over-dot denotes the derivative with respect to $c\tau$: $\cdot = \mathrm{d}/c\mathrm{d}\tau = \mathrm{d}/b(t)c\mathrm{d}t$, and ζ is defined in Eq. (113). Since the matter energy density ε is determined by Eq. (132), the two unknown functions $a(\tau)$, $b(\tau)$ are fully determined by the two equations (137), (138). In the finite-range cosmology, in contrast to GR, the function b is not arbitrary.

A direct consequence of Eqs. (137), (138) has the form of $\dot{M}^0_0 + 3(\dot{a}/a)[M^0_0 - M^1_1] = 0$. This is the single nonvanishing equation from the set of equations (26). The left-hand-side of this equation can be transformed to a total time-derivative. This fact can be seen more easily from the equivalent form of this equation, stemming from Eq. (27):

$$\mathcal{M}^{00}{}_{;0} = \frac{3}{16}\frac{\alpha^2}{\zeta+2}\frac{d}{dt}\left[3\frac{a^6}{b^2} - (1-4\zeta)a^2b^2 - 2(2\zeta+1)a^4 + 8\zeta\frac{a^3}{b} - 8\zeta\right] = 0. \tag{139}$$

Equation (139) says that the combination of terms in the square brackets must be a constant. The value of this constant is determined by the observation that the zero gravitational field, i.e. $h^{\mu\nu} = 0$ and, hence, $b = 1/a$, $a^2 = \pm 1$, should also be a solution of this equation. On this ground, one finds that the integration constant should be equal to zero. As a result, Eq. (139) yields to the following algebraic relationship between a and b:

$$3a^6 - a^2b^4 - 2a^4b^2 + 4\zeta(a^2b^4 - a^4b^2 + 2a^3b - 2b^2) = 0. \tag{140}$$

In principle, this equation allows one to express b in terms of a for arbitrary ζ. Then, the only differential equation to be solved is one of the two equations (137), (138); say, the first one. Although this strategy solves the cosmological problem in principle, it is not easy to implement it analytically, for arbitrary ζ. This is why we shall concentrate on particular simplifying choices of the parameter ζ.

The first interesting case is $\zeta = 0$. According to the definition (113), this case corresponds to $\beta^2 = 0$ and, hence, to the zero mass of the *spin* $-$ 0 graviton. If $\zeta = 0$, Eq. (140) requires $b = \pm a$. Then, both, M^0_0 and M^1_1 vanish identically, and equations (137), (138) retain their GR form. Thus, in the case of $\zeta = 0$, the finite-range cosmology is exactly the same as the GR cosmology, independently of the mass of the *spin* $-$ 2 graviton. Therefore, deviations from the GR cosmology can arise only if the parameter β is non-zero.

Introducing

$$y = \frac{a}{b},$$

one can rewrite Eq. (140) in the form

$$y^4 + \frac{8\zeta}{3a^2}y^3 - \frac{2}{3a^4}\left(a^4 + 2\zeta a^4 + 4\zeta\right)y^2 + \frac{1}{3}(4\zeta - 1) = 0. \tag{141}$$

This equation helps one to identify one more special case: $\zeta = 1/4$. In this case, the 4th-order algebraic equation (141) reduces to the 2nd-order equation, with the solution

$$y \equiv \frac{a}{b} = \frac{-1 + \sqrt{7 + 9a^4}}{3a^2}. \tag{142}$$

(We have eliminated one of solutions by demanding $y \geq 0$.) We shall study this case analytically and numerically in considerable details. However, we will also be presenting, whenever possible, more general relationships, valid for $\zeta \neq 1/4$. In terms of the function y, equation (137) can be written as

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{8}\frac{\alpha^2}{\zeta+2}\left[y^3 - (1-4\zeta)\frac{1}{y} + \frac{2\zeta}{a^2}\left(y^2 - 3\right)\right] = \frac{\kappa\varepsilon_0}{a^{3(q+1)}}\ , \tag{143}$$

C. The early-time and the late-time evolution in the finite-range cosmology

Equation (142) demonstrates that, in the massive gravity, the numerical value of $a(\tau)$ becomes important. The asymptotic formulas for $y(a^2)$ and, hence, for $M^0_0(a^2)$, depend on whether $a^2 \gg 1$ or $a^2 \ll 1$. In the former limit, the approximate expression of Eq. (142) is

$$y \approx 1, \qquad a^2 \gg 1.$$

This approximate solution is valid for any ζ. Then, the approximate expression for M^0_0 is

$$M^0_0 \approx \frac{3}{2(\zeta+2)}\beta^2, \qquad a^2 \gg 1.$$

In the latter limit, the approximate expression of Eq. (142) is

$$y \approx \frac{1}{3a^2}\left(-1+\sqrt{7}\right), \qquad a^2 \ll 1.$$

The generalization of this solution to $\zeta \neq 1/4$ is

$$y \approx \frac{4\zeta}{3a^2}\left(-1+\sqrt{1+\frac{3}{2\zeta}}\right), \qquad a^2 \ll 1,$$

and we ignore other solutions to Eq. (141) in this limit. Then, the approximate expression for M^0_0 ($\zeta = 1/4$) in the limit of small a^2 is

$$M^0_0 \approx \frac{2(7\sqrt{7}-10)}{81}\frac{\beta^2}{a^6}, \qquad a^2 \ll 1.$$

The full behaviour of M^0_0 ($\zeta = 1/4$) as a function of a^2 is described by a smooth curve that descends as $M^0_0 \propto a^{-6}$ from +infinity at the origin, then comes to the minimum, equal to zero, at $a^2 = 1$, and rises again to reach asymptotically, for $a^2 \to \infty$, the constant level $M^0_0 = 2\beta^2/3$. This behaviour is illustrated in a numerical plot of Fig. 9.

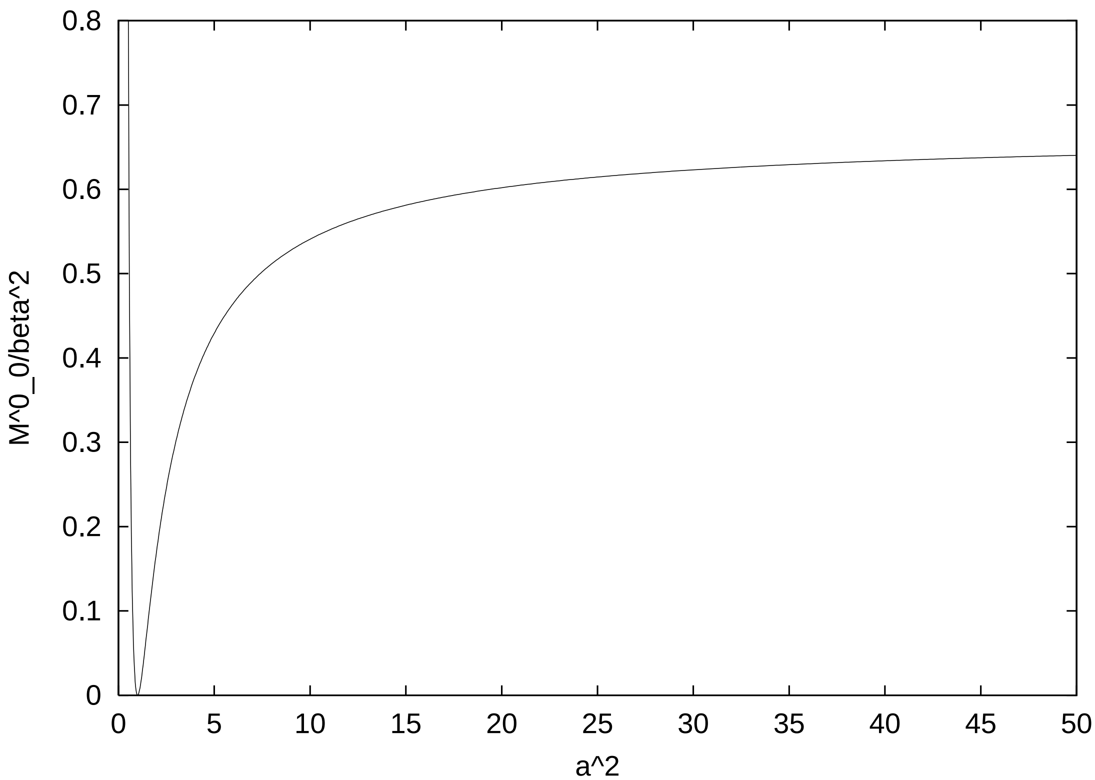

FIG. 9. $M^0{}_0/\beta^2$ as a function of a^2 for $\zeta = 1/4$.

We shall first show that there is a long interval of evolution, when the finite-range cosmology is practically indistinguishable from the GR cosmology. In addition to the already defined scales $l_H(\tau) = a/\dot{a}$ and $l_0 = 1/\sqrt{\kappa\varepsilon_0}$, we introduce the finite-range scale $l_\beta = 1/\beta$. For simplicity, we will be confined to the case $\zeta = 1/4$, so that the finite-range scale associated with α is simply related to the introduced one: $l_\alpha = 1/\alpha = l_\beta/2$. Let us consider the interval of evolution, when $a^2 \gg 1$, but the massive term can be neglected. If $a^2 \gg 1$, the second (massive) term in Eq. (143) is $M^0_0 \approx 2/3l^2_\beta$. The first term is $3/l^2_H$, so when $l_H(\tau) \ll l_\beta$, the massive term can be neglected in comparison with the first term. Then, the first term is balanced by the right-hand-side of Eq. (143), i.e. by the term $1/l_0^2 a^{3(q+1)}$. From the comparison of these two terms one finds that the inequality $l_0 \ll l_H(\tau)$ must hold. Thus, in the interval of evolution such that $l_0 \ll l_H(\tau) \ll l_\beta$, the $a(\tau)$ and $\varepsilon(\tau)$ are well approximated by their GR expressions, Eq. (134) and Eq. (135).

We now turn to cosmological evolution at early times, when $a^2 \ll 1$. In GR, the early evolution begins with the singularity $a(\tau) = 0$ at $\tau = 0$. The scale factor cannot go through a regular minimum, where $\dot{a} = 0$ and $a = a_{min}$. If it were possible that $\dot{a} = 0$ at some moment of time, the l.h.s. of Eq. (133) would vanish at that moment of time, while the r.h.s. is strictly positive, so that Eq. (133) would not be satisfied. The situation changes in the finite-range gravity. It follows from Eq. (143) that, in contrast to GR, $a(\tau)$ cannot be arbitrarily small. Indeed, if it were possible that $a^2 \to 0$, the r.h.s. of Eq. (143), which is proportional to $1/a^{3(q+1)}$, would be negligibly small in comparison with the massive term $M^0_{\ 0}$, which grows as $\sim 1/a^6$. But then, the two positive terms in Eq. (143), the first and the second one, would not be able to balance each other, and Eq. (143) would not be satisfied. Instead, the scale factor of the finite-range cosmology goes through a regular minimum, where $\dot{a} = 0$. Near the minimum, the Hubble radius tends to infinity, so that $l_H \gg l_\beta$. The first term in Eq. (143) can be neglected in comparison with the second (massive) term. From the comparison of the massive term with the r.h.s. of Eq. (143), one can evaluate the minimum value a_{min} of the scale factor:

$$a_{min} \approx \left(\frac{l_0}{l_\beta}\right)^{\frac{2}{3(1-q)}} .$$

Since $l_0 \ll l_\beta$ and the exponent $2/3(1-q)$ is strictly positive, we see that $a_{min} \ll 1$ as it should be. In the particular case of the early radiation-dominated era, i.e. for $q = 1/3$, one finds that $a_{min} \approx l_0/l_\beta$. The minimum of $a(\tau)$ is deeper for larger values of l_β and, hence, for smaller values of the mass m_0 of the $spin-0$ graviton. (Certainly, the expected deep minimum of $a(\tau)$ does not invalidate the quantum-mechanical generation of cosmological perturbations and their observational consequences [44].) The vicinity of the minimum is shown in Fig. 10 as a numerical solution to Eq. (143) for $\zeta = 1/4$, $\alpha^2 l_0^2 = 10^{-2}$, and the initial data $a = 1$ at $c\tau/l_0 = \sqrt{3}/2$. The value of the parameter $\alpha^2 l_0^2$ is taken large, because otherwise the graph would be superimposed on the Friedmann solution.

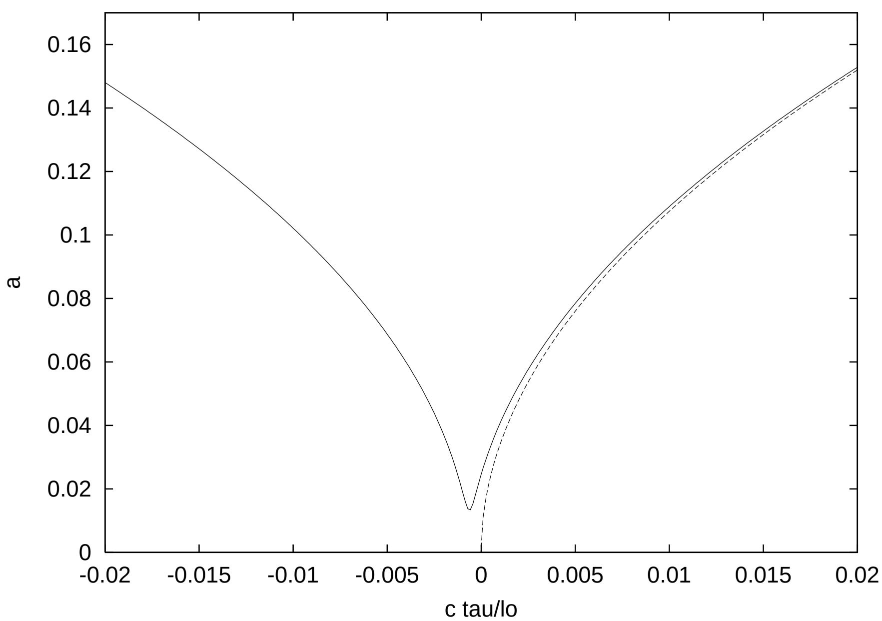

FIG. 10. The dashed line is a Friedmann solution with initial data $a = 1$ at $c\tau/l_0 = \sqrt{3}/2$. The solid line is a numerical solution to Eq. (143) for $\zeta = 1/4, \alpha^2 l_0^2 = 10^{-2}$ and the same initial data. Both solutions are found for $q = 1/3$.

We shall now consider the late-time evolution in the finite-range cosmology. First, we note that, in contrast to GR, the scale factor $a(\tau)$ cannot grow indefinitely. Indeed, if it were possible that $a(\tau) \to \infty$, the r.h.s. of Eq. (143) would be negligibly small in comparison with the constant massive term $M^0_{\ 0} \sim 1/l_\beta^2$. But then, the two positive terms on the l.h.s. of Eq. (143) would not be able to balance each other. Instead, the scale factor $a(\tau)$ goes through a regular maximum, where $\dot{a} = 0$ and $a = a_{max}$. Near the maximum, the Hubble radius tends to infinity, and, similar to what takes place in the vicinity of the regular minimum, one has $l_H \gg l_\beta$. The first term in Eq. (143) can be neglected, and from the comparison of the massive term with the r.h.s. of Eq. (143) one can evaluate the maximum value a_{max} of the scale factor:

$$a_{max} \approx \left(\frac{l_\beta}{l_0}\right)^{\frac{2}{3(1+q)}} .$$

Since $l_0 \ll l_\beta$ and the exponent $2/3(1+q)$ is strictly positive, one has $a_{max} \gg 1$, as it should be. In the particular case of the late matter-dominated era, i.e. for $q = 0$, one finds that $a_{max} \approx (l_\beta/l_0)^{2/3}$. The maximum of $a(\tau)$ is higher for larger values of l_β and, hence, for smaller values of the mass m_0 of the *spin* $- 0$ graviton. The energy density at the maximum of expansion is given by the universal (valid for any q) formula $\kappa\varepsilon_{max} \approx 1/l_\beta^2$. The late-time behaviour of $a(\tau)$ admits an analytical treatment. At $a^2 \gg 1$, the approximate form of Eq. (143) is

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\beta^2}{2(\zeta+2)} = \frac{1}{l_0^2 a^3} , \tag{144}$$

where we have taken $q = 0$. This equation can be rearranged to read

$$\frac{l_0\sqrt{a}\mathrm{d}a}{\sqrt{1-\frac{3l_0^2\beta^2}{2(\zeta+2)}a^3}} = \frac{c\mathrm{d}\tau}{\sqrt{3}}.$$

The exact solution to Eq. (144) takes the form

$$\frac{1}{\sqrt{3}}c(\tau+\tau_0) = \begin{cases} \sqrt{\frac{2(\zeta+2)}{3\beta^2}}\arcsin\left[a^{3/2}\sqrt{\frac{3l_0^2\beta^2}{2(\zeta+2)}}\right] , & \text{if } \beta^2 > 0. \\ \sqrt{\frac{2(\zeta+2)}{-3\beta^2}}\ln\left[a^{3/2}\sqrt{\frac{-3l_0^2\beta^2}{2(\zeta+2)}} + \sqrt{\frac{-3l_0^2\beta^2}{2(\zeta+2)}a^3+1}\right] , & \text{if } \beta^2 < 0. \end{cases} \tag{145}$$

Let us start the analysis of Eq. (145) from the usual case $\beta^2 > 0$. The scale factor at late times can be written as

$$a(\tau) \approx \left(\frac{l_\beta}{l_0}\right)^{2/3}\left(\frac{2(\zeta+2)}{3}\right)^{1/3}\sin^{2/3}\left[\frac{c(\tau+\tau_0)}{l_\beta\sqrt{2(\zeta+2)}}\right] .$$

It is clear from this formula that $a(\tau)$ goes through the regular maximum at the moment of time when the argument of the *sine* function reaches $\pi/2$. In agreement with the evaluations done above, the maximum value of the scale factor is given by $a_{max} \approx (l_\beta/l_0)^{2/3}$. On the other hand, as we have already shown, the scale factor experiences also a regular minimum. Thus, in a quite remarkable manner, the arbitrarily small mass-terms (9) make the cosmological evolution oscillatory. [4] The minima and maxima of $a(\tau)$ are "turning points" of the effective potential in Eq. (143), which consists of M_0^0 and the r.h.s. of that equation.

We now turn to the case $\beta^2 < 0$. As was explained in Introduction, the interpretation of parameters α^2 and β^2 in terms of masses requires them to be positive. However, the massive Lagrangian itself allows the parameters α^2 and β^2 to be negative. It is interesting to note that the case $\beta^2 < 0$ makes the scale factor $a(\tau)$ exponentially growing with τ at very late times. Indeed, for $a^3 \gg 2(\zeta+2)/(-3l_0^2\beta^2)$, the second line of Eq. (145) can be rearranged to read:

$$a(\tau) \approx \left(\frac{\zeta+2}{-6l_0^2\beta^2}\right)^{1/3} e^{c(\tau+\tau_0)\sqrt{\frac{-2\beta^2}{9(\zeta+2)}}} .$$

The part of evolution, where $a(\tau)$ experiences an exponential growth with τ, mimics the contribution of the positive cosmological Λ-term. This evolution is also similar to the "accelerated expansion" driven (in framework of GR) by some speculative forms of matter known as "quintessence" and "dark energy". In contrast to these possibilities, the finite-range gravity provides for the "accelerated expansion" of the Universe at the expense of a specific modification of GR, without resorting to exotic forms of matter. The possible behaviour of the scale factor, under the assumption

[4] It is likely that, sooner or later, this behaviour of $a(\tau)$ will be declared a prediction of "inflation"; we are assured by previous inflationary literature that everything in life was predicted by inflation or, at most, by alternative to inflation.

that $\beta^2 < 0$, is illustrated in Fig. 11. (Modifications to cosmological evolution caused by various alternative theories of gravity have been discussed in references [40], [41], [45], [46], [47].)

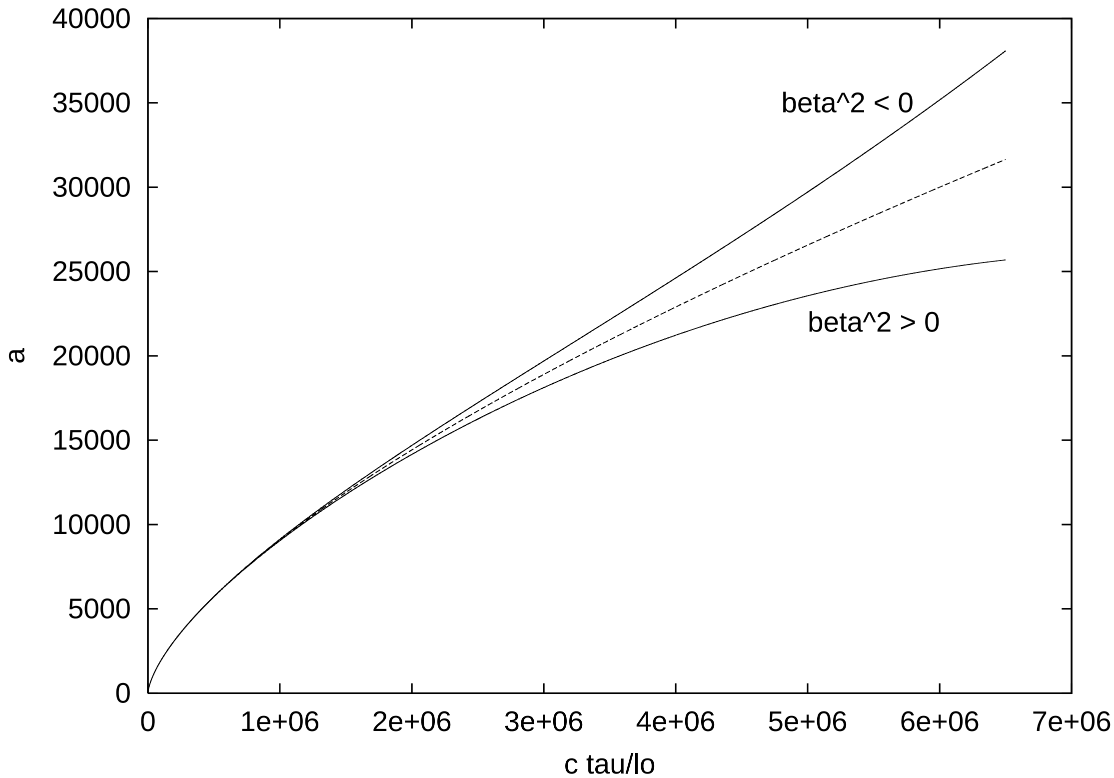

FIG. 11. The dashed line is a Friedmann solution of GR, the upper solid line is a solution with $\beta^2 < 0$ and the lower solid line is a solution with $\beta^2 > 0$. The graphs are calculated for $q = 0$, $\zeta = 1/4$, $\alpha^2 l_0^2 = 10^{-12}$ and the initial data $a = 1$ at $c\tau/l_0 = 2/\sqrt{3}$.

For simplicity, we were considering here a conventional one-component matter. If, nevertheless, nature does allow for a cosmological Λ-term (of whatever origin, sign and value) as well as for various types of exotic matter, then, of course, the effects of massive gravity at early and late times can be partially or totally compensated by the Λ-term or exotic matter. In this case, the number of possibilities for cosmological evolution and its explanation increases greatly. For instance, the dynamical effect of a huge positive Λ-term could be almost compensated by an appropriate massive term with $\beta^2 > 0$, with the net result of a modest "accelerated expansion" in the present era. Clearly, more definitive cosmological observations are badly needed.

APPENDIX A

Here we will prove the equivalence of Eq. (26) and Eq. (27). First, we can rewrite Eq. (26) in a more convenient for our purposes form:

$$(\sqrt{-g}M^{\mu}{}_{\nu})_{|\mu} \equiv (\sqrt{-g}M^{\mu}{}_{\nu})_{,\mu} - \Gamma^{\sigma}{}_{\mu\nu}(\sqrt{-g}M^{\mu}{}_{\sigma}) = 0. \tag{A1}$$

Using $m_{\alpha\beta} \equiv 2(k_1 h_{\alpha\beta} + k_2 \gamma_{\alpha\beta} h)$ in Eq. (22), we have

$$M^{\mu}_{\nu} = (g^{\mu\alpha}\delta^{\beta}_{\nu} - \frac{1}{2}g^{\alpha\beta}\delta^{\mu}_{\nu})m_{\alpha\beta}.$$

Then,

$$\sqrt{-g}M^{\mu}_{\nu} = \sqrt{-\gamma}\left[(\gamma^{\mu\alpha} + h^{\alpha\beta})\delta^{\beta}_{\nu} - \frac{1}{2}(\gamma^{\alpha\beta} + h^{\alpha\beta})\delta^{\mu}_{\nu}\right] m_{\alpha\beta}.$$

The last expression is a tensor density with respect to the metric $\gamma_{\alpha\beta}$, so we can rewrite Eq.(A1) as follows:

$$(\sqrt{-g}M^\mu{}_\nu)_{|\mu} = (\sqrt{-g}M^\mu{}_\nu)_{;\mu} - (\Gamma^\sigma{}_{\mu\nu} - C^\sigma{}_{\mu\nu})(\sqrt{-g}M^\mu{}_\sigma) = 0. \tag{A2}$$

Consider specifically the term $\Gamma^\sigma{}_{\mu\nu}\sqrt{-g}M^\mu{}_\sigma$. Writing $\Gamma^\sigma{}_{\mu\nu}$ explicitely as a function of $g^{\mu\nu}$, we obtain

$$\Gamma^\sigma{}_{\mu\nu}\sqrt{-g}M^\mu{}_\sigma = \frac{1}{2}\sqrt{-g}\left(g^{\mu\alpha}\delta^\beta_\nu - \frac{1}{2}g^{\alpha\beta}\delta^\mu_\nu\right)g^{\sigma\omega}(g_{\omega\mu,\nu} + g_{\omega\nu,\mu} - g_{\mu\nu,\omega})m_{\alpha\beta}. \tag{A3}$$

Using $\sqrt{-g}g^{\mu\alpha}g^{\beta\omega}g_{\omega\mu,\nu} = (\sqrt{-g}g^{\alpha\beta})_{,\nu} + \frac{1}{2}g^{\alpha\beta}g_{\sigma\omega}(\sqrt{-g}g^{\sigma\omega})$ and $\sqrt{-g}g^{\sigma\omega}g_{\sigma\omega,\nu} = g_{\sigma\omega}(\sqrt{-g}g^{\sigma\omega})_{,\nu}$, Eq.(A3) can be transformed to

$$\Gamma^\sigma{}_{\mu\nu}\sqrt{-g}M^\mu{}_\sigma = -\frac{1}{2}(\sqrt{-g}g^{\alpha\beta})_{,\nu}m_{\alpha\beta}. \tag{A4}$$

The term $\sqrt{-g}g^{\alpha\beta}$ is again a tensor density as seen from Eq. (4). So we can trade the ordinary derivative of this term for the covariant one according to:

$$(\sqrt{-g}g^{\alpha\beta})_{,\nu} = (\sqrt{-g}g^{\alpha\beta})_{;\nu} - \sqrt{-g}g^{\alpha\sigma}C^\beta{}_{\sigma\nu} - \sqrt{-g}g^{\beta\sigma}C^\alpha{}_{\sigma\nu} + \sqrt{-g}g^{\alpha\beta}C_\nu.$$

Using this expression in (A4) and substituting (A4) into the last term of (A2), we get

$$\sqrt{-g}\left(g^{\mu\alpha}\delta^\beta_\sigma - \frac{1}{2}g^{\alpha\beta}\delta^\mu_\sigma\right)(\Gamma^\sigma{}_{\mu\nu} - C^\sigma{}_{\mu\nu})m_{\alpha\beta} = -\frac{1}{2}(\sqrt{-g}g^{\alpha\beta})_{;\nu}m_{\alpha\beta} = \sqrt{-\gamma}h^{\alpha\beta}{}_{;\nu}m_{\alpha\beta}. \tag{A5}$$

The final step is to use the explicit form of $m_{\alpha\beta}$ and $\sqrt{-\gamma}h^{\alpha\beta}{}_{;\nu}m_{\alpha\beta} = \sqrt{-\gamma}(k_1h^{\alpha\beta}h_{\alpha\beta} + k_2h^2)_{;\nu}$ in (A5) and (A2), which leads us to the desired result:

$$\sqrt{-g}M^\mu{}_{\nu|\mu} = \sqrt{-\gamma}\left[2k_1h^\mu{}_\nu - \delta^\mu_\nu(k_1 + 2k_2)h + 2k_1h^{\mu\alpha}h_{\nu\alpha} + 2k_2h^\mu{}_\nu h - \frac{1}{2}\delta^\mu_\nu(k_1h^{\alpha\beta}h_{\alpha\beta} + k_2h^2)\right]_{;\mu} = 0 \tag{A6}$$

One can see that the last equation proves the equivalence of Eq. (26) and Eq. (27).

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